ARTICLE IN PRESS
Physics Reports ▮ (▮▮▮▮) ▮▮▮–▮▮▮

Contents lists available at ScienceDirect

Physics Reports

journal homepage: www.elsevier.com/locate/physrep

Quantum coherence and geometric quantum discord

Ming-Liang Hu [a,b], Xueyuan Hu [c], Jieci Wang [a,d], Yi Peng [a], Yu-Ran Zhang [e,f,a], Heng Fan [a,g,h,*]

[a] *Institute of Physics, Chinese Academy of Sciences, Beijing 100190, China*
[b] *School of Science, Xi'an University of Posts and Telecommunications, Xi'an 710121, China*
[c] *School of Information Science and Engineering, Shandong University, Jinan 250100, China*
[d] *Department of Physics, and Synergetic Innovation Center for Quantum Effects, Hunan Normal University, Changsha, Hunan 410081, China*
[e] *Beijing Computational Science Research Center, Beijing 100094, China*
[f] *Theoretical Quantum Physics Laboratory, RIKEN Cluster for Pioneering Research, Wako-shi, Saitama 351-0198, Japan*
[g] *CAS Center for Excellence in Topological Quantum Computation, University of Chinese Academy of Sciences, Beijing 100190, China*
[h] *Collaborative Innovation Center of Quantum Matter, Beijing 100190, China*

ARTICLE INFO

Article history:
Accepted 31 July 2018
Available online xxxx
Editor: G Refael

ABSTRACT

Quantum coherence and quantum correlations are of fundamental and practical significance for the development of quantum mechanics. They are also cornerstones of quantum computation and quantum communication theory. Searching physically meaningful and mathematically rigorous quantifiers of them are long-standing concerns of the community of quantum information science, and various faithful measures have been introduced so far. We review in this paper the measures of discordlike quantum correlations for bipartite and multipartite systems, the measures of quantum coherence for any single quantum system, and their relationship in different settings. Our aim is to provide a full review about the resource theory of quantum coherence, including its application in many-body systems, and the discordlike quantum correlations which were defined based on the various distance measures of states. We discuss the interrelations between quantum coherence and quantum correlations established in an operational way, and the fundamental characteristics of quantum coherence such as their complementarity under different basis sets, their duality with path information of an interference experiment, their distillation and dilution under different operations, and some new viewpoints of the superiority of the quantum algorithms from the perspective of quantum coherence. Additionally, we review properties of geometric quantum correlations and quantum coherence under noisy quantum channels. Finally, the main progresses for the study of quantum correlations and quantum coherence in the relativistic settings are reviewed. All these results provide an overview for the conceptual implications and basic connections of quantum coherence, quantum correlations, and their potential applications in various related subjects of physics.

Contents

* Corresponding author at: Institute of Physics, Chinese Academy of Sciences, Beijing 100190, China.
E-mail addresses: jcwang@hunnu.edu.cn (J. Wang), hfan@iphy.ac.cn (H. Fan).

https://doi.org/10.1016/j.physrep.2018.07.004

1. Introduction

Quantum correlations and quantum coherence are two fundamental concepts in quantum theory (Nielsen and Chuang, 2000). While quantum correlations characterize the quantum features of a bipartite or multipartite system, quantum coherence was defined for the integral system. Despite this obvious difference and as the different embodiments of the unique characteristics of a quantum system, they are also intimately related to each other which can be scrutinized from different perspectives. Moreover, from a practical point of view, quantum correlations and quantum coherence are also invaluable physical resources for quantum information and computation tasks (Nielsen and Chuang, 2000), hence remain research focuses since the early days of quantum mechanics.

The quantum correlation in a system can be characterized and quantified from different perspectives. Historically, there are various categories of quantum correlation measures being proposed, prominent examples include the widely-studied Bell-type nonlocal correlation (Genovese, 2005) and quantum entanglement (Horodecki et al., 2009). At the beginning of this century, a new framework for quantifying quantum correlations was formulated by Henderson and Vedral (2001), as well as by Ollivier and Zurek (2001) in the study of quantum discord (QD). Within this framework, an abundance of discordlike quantum correlation measures were proposed and studied from different aspects in the past years (Modi et al., 2012; Adesso et al., 2016).

Quantum coherence, another embodiment of the superposition principle of quantum states, is essential for many novel and intriguing characteristics of quantum systems (Ficek and Swain, 2005). It is also of equal importance as quantum correlations in the studies of both bipartite and multipartite systems (Modi et al., 2012). Constructing a mathematically rigorous and physically meaningful framework for its characterization and quantification was a main pursue of researchers in quantum community, as this is not only essential for quantum foundations, but can also provide the basis for its potential applications in a wide variety of promising subjects, such as quantum thermodynamics (Lostaglio et al., 2015a; Narasimhachar and Gour, 2015; Lostaglio et al., 2015b; Gour et al., 2015), reference frames (Bartlett et al., 2007), and quantum biology (Lambert et al., 2013).

Coherence is a main concern of quantum optics historically, and it was usually studied from the perspective of phase space distributions or multipoint correlations. Instead of reviewing these approaches, we focus on the recent developments about quantitative characterization of coherence (Baumgratz et al., 2014), the essence of which is the adoption of the viewpoint of treating quantum coherence as a physical resource, just like the resource theory of entanglement. A big advantage of this approach is that the related resource theory of entanglement can be adapted for introducing similar measures of quantum coherence (Horodecki et al., 2009).

The geometric characterization of quantum correlations in a bipartite system has been studied extensively in the past five years, with the corresponding measures being defined based on various (pseudo) distance measures of two states. The central idea for this kind of definition is that the distance between the considered state and the state without the desired property is a measure of that property (Modi et al., 2010). The similar idea can also be used to characterize quantum coherence of a system, and along this line, many quantum coherence measures have been proposed. We will review these coherence measures and show their interrelations with quantum correlations. Moreover, we will also discuss the other major progresses achieved in studying fundamental problems of geometric quantum correlations and quantum coherence, their dynamics under noisy quantum channels, and their applications in the study of many-body systems.

During the preparation of this review, we became aware of another nice review work by Streltsov and Adesso (2017a). Though seemingly discussing a similar topic, our concerns are different. Streltsov and Adesso (2017a) reviewed the resource theory of coherence, while our concerns are quantum coherence, quantum correlations, and their intrinsic connections. In our work, besides presenting a comprehensive view of the main developments of quantum coherence and quantum correlations, we try to summarize and reformulate some calculations scattering in a large number of literature. In particular, we have also reviewed in detail the main progresses for the study of quantum correlations and quantum coherence in the relativistic settings. We hope that this review may be useful for both beginners and seniors in quantum information science.

This review is organized as follows. In Section 2, we first briefly recall the framework for defining QD and the seminal measure of QD defined via the discrepancy between quantum mutual information (QMI) and the classical aspect of correlations. Then, we present in detail the recently introduced geometric quantum discords (GQDs) and measurement-induced nonlocality (MIN) defined by virtue of various distance measures of quantum states, e.g., the Hilbert–Schmidt (HS) norm, the trace norm, the Bures distance, the Hellinger distance, the relative entropy, and the Wigner–Yanase (WY) skew information. We will also present a review concerning the fundamental connections among these correlation measures and the recent developments about their potential role in typical tasks of quantum information processing.

In Section 3, we first review the basic aspects related to the resource theory of quantum coherence including structure of the corresponding free states and free operations. We then present a detailed review on those recent developments of the quantification of quantum coherence. They can be classified roughly into the following categories: the distance-based measure of coherence, the entanglement-based measure of coherence, the convex roof measure of coherence, the robustness of coherence, the Tsallis relative entropy measure of coherence, and the skew-information-based measure of coherence. For every category, there are also different definitions of coherence measures. Moreover, some of their extensions valid for infinite-dimensional systems and some other coherence measures defined within slightly different frameworks, for example, the different construction of free operations, will also be reviewed in this section.

In Section 4, we review recent developments on interpretations of the aforementioned quantum coherence measures. First, we show from an operational perspective that quantum coherence is intimately related to quantum correlations in bipartite and multipartite systems, highlighting the fundamental position of coherence in quantum theory. Then, we review the complementarity relations of quantum coherence under both mutually unbiased bases (MUBs) and incompatible bases, as well as the complementarity relation between coherence and mixedness and between coherence and path distinguishability. Finally, the distillation and dilution of quantum coherence with different forms of restricted operations and the average coherence of randomly sampled states are discussed.

In Section 5, we consider the role of quantum coherence in some quantum information processing tasks, including quantum state merging, Deutsch–Jozsa algorithm, Grover search algorithm, and deterministic quantum computation with one qubit (DQC1). The role of coherence in the quantum metrology tasks such as phase discrimination and subchannel discrimination is also reviewed. Of course, as quantum coherence underlies different forms of quantum correlations which are essential for quantum information, we focus here only on those of the closely related topics.

In Section 6, we discuss dynamics of quantum coherence and QD, mainly concentrating on their singular behaviors in open quantum systems. In this section, we first review frozen phenomena of quantum coherence and QD which are preferable for information processing tasks. Then, we discuss potential ways for protecting and enhancing quantum coherence and QD. Two closely related problems, i.e., the resource creating and breaking powers of quantum channel and the factorization relation for the evolution equation of QD and quantum coherence are described in detail.

In Section 7, we consider quantum coherence in explicit physical systems. We employ the various spin-chain model to show that quantum coherence can be used to study the long-range order, valence-bond-solid states, localized and thermalized states, and quantum phase transitions of the many-body systems. This shows that the resource theory of quantum coherence is not only of fundamental but is also of practical significance.

In Section 8, we present a review on recent progresses of quantum coherence and QD in relativistic settings, including their behaviors for the free field modes, for curved spacetime and expanding universe, and for noninertial cavity modes. We

Please cite this article in press as: Hu M.-L., et al., Quantum coherence and geometric quantum discord. Physics Reports (2018), https://doi.org/10.1016/j.physrep.2018.07.004.

provide a summary of the Unruh temperature, Hawking temperature, expansion rate of the universe, accelerated motion of cavities and detectors, and boundary conditions of the field on quantum correlations and quantum coherence. Quantum correlations for particle detectors and the dynamical Casimir effects on these correlations are also provided here.

Finally, in Section 9, we present a concluding remark on the main results of this review. We hope the review be helpful for further exploration in these fields. Several open questions are also raised for possible future research.

2. Geometric quantum correlation measures

The widely used discordlike quantum correlation measures proposed in the past ten years can be categorized roughly into two different families, namely, those based on the entropy theory, and those based on various distance measures of quantum states. A detailed overview of the first category has already been given by Modi et al. (2012), and there are no new measures being proposed along this line in recent years, so we will recall only the original definition of QD and several of the related entropic measures for self consistency of this review. Our main concern will be the second category of discordlike quantum correlation measures. Most of them are proposed after the year 2012, and have been proven to be well defined. The related notions and approaches used in their definitions have also been proven useful for introducing coherence measures. In particular, this allows us to put the discordlike correlations and quantum coherence on an equal footing, which facilitates one's investigation of the interrelation between these two different quantifiers of quantumness. Moreover, as the definitions of GQD and quantum coherence have something in common, the well developed methods for calculating GQD are also enlightening for deriving analytical expressions of the related coherence measures.

To begin with, we recall the concept of QD that was framed from the viewpoints of information theory. Within the seminal framework formulated by Henderson and Vedral (2001) as well as Ollivier and Zurek (2001), it was defined based on the partition of the total correlation in state ρ_{AB} into two different parts, that is, the classical part and the quantum part. The QMI $I(\rho_{AB})$ was used as a measure of total correlation, and it reads

$$I(\rho_{AB}) = S(\rho_A) + S(\rho_B) - S(\rho_{AB}), \tag{1}$$

with $S(\rho_X) = -\mathrm{tr}(\rho_X \log_2 \rho_X)$ ($X = A, B$, or AB) being the von Neumann entropy.

Similar to most correlation measures whose quantification implies measurement, the classical correlation was also defined from a measurement perspective. Henderson and Vedral (2001) proposed four defining conditions for a classical measure $J(\rho_{AB})$: (i) $J(\rho_{AB}) = 0$ for any $\rho_{AB} = \rho_A \otimes \rho_B$, (ii) it should be locally unitary invariant, (iii) It is nonincreasing under local operations, and (iv) $J(\rho_{AB}) = S(\rho_A) = S(\rho_B)$ for pure states. Based on these conditions, they defined the classical correlation as the maximum information about one party (say B) of a bipartite system that can be extracted by performing the positive operator valued measure (POVM) on the other party (say A). If the POVM $\{E_k^A\}$ with elements $E_k^A = M_k^{A\dagger} M_k^A$ is performed on party A, one can obtain the postmeasurement state of the system and the conditional state of party B as

$$\rho'_{AB} = \sum_k M_k^A \rho_{AB} M_k^{A\dagger}, \quad \rho_{B|E_k^A} = \frac{\mathrm{tr}_A(E_k^A \rho_{AB})}{p_k}, \tag{2}$$

where $p_k = \mathrm{tr}(E_k^A \rho_{AB})$ is the probability for obtaining the outcome k. The classical correlation is given by

$$J_A(\rho_{AB}) = S(\rho_B) - \min_{\{E_k^A\}} S(B|\{E_k^A\}), \tag{3}$$

where

$$S(B|\{E_k^A\}) = \sum_k p_k S(\rho_{B|E_k^A}), \tag{4}$$

is the averaged conditional entropy of the postmeasurement state ρ'_{AB}.

The QD is then defined by the discrepancy between $I(\rho_{AB})$ and $J_A(\rho_{AB})$ as

$$D_A(\rho_{AB}) = I(\rho_{AB}) - J_A(\rho_{AB}) = \min_{\{E_k^A\}} S(B|\{E_k^A\}) - S(B|A), \tag{5}$$

where $S(B|A) = S(\rho_{AB}) - S(\rho_A)$ is the conditional entropy. Of course, one can also define $D_B(\rho_{AB})$ by performing the measurements on party B. In general, $D_A(\rho_{AB}) \neq D_B(\rho_{AB})$, that is, the QD is an asymmetric quantity.

The QD defined above is based on POVM, but the corresponding maximization is generally a notoriously challenging task. So sometimes one can also consider the set of rank-one projectors $\{\Pi_k^A\}$, for which the postmeasurement state turns out to

$$\rho'_{AB} = \sum_k p_k \Pi_k^A \otimes \rho_{B|\Pi_k^A}. \tag{6}$$

This yields $I(\rho'_{AB}) = S(\rho_B) - \sum_k p_k S(\rho_{B|\Pi_k^A})$, and the QD becomes

$$D_A(\rho_{AB}) = I(\rho_{AB}) - \max_{\{\Pi_k^A\}} I(\rho'_{AB}). \tag{7}$$

Hence the intuitive meaning of QD can be interpreted as the minimal loss of correlations due to the local projective measurements $\{\Pi_k^A\}$. Indeed, it suffices to consider only the projective measurements $\{\Pi_k^A\}$ for two-qubit states in most cases (Galve et al., 2011; Hamieh et al., 2004). Moreover, the optimal measurement strategy (over four-element POVM) for obtaining classical correlation $J_A(\rho_{AB})$ in many-body system has been studied by Amico et al. (2012).

Similarly, a symmetric version of QD was also proposed. It reads (Wu et al., 2009; Girolami et al., 2011)

$$D_s(\rho_{AB}) = I(\rho_{AB}) - \max_{\{\Pi_k^A\otimes\Pi_l^B\}} I(\rho''_{AB}), \tag{8}$$

where $\rho''_{AB} = \sum_k p_{kl}\Pi_k^A \otimes \otimes \Pi_l^B$.

Apart from the above QD measure, Zurek (2003) presented a slightly different discordlike correlation measure which was called thermal QD, and is defined as

$$\tilde{D}_A(\rho_{AB}) = \min_{\{\Pi_k^A\}}[S(\rho'_A) + S(B|\{\Pi_k^A\})] - S(\rho_{AB}), \tag{9}$$

where ρ'_A is the reduced state of ρ'_{AB} in Eq. (6), and $S(\rho'_A) = H(\{p_k\})$, with $H(\{p_k\})$ being the Shannon entropy function and the probability $p_k = \mathrm{tr}(\Pi_k^A\rho_{AB}\Pi_k^A)$.

Apart from the above entropic measure and the other related entropic measures summarized in detail by Modi et al. (2012), QD can also be measured from a geometric aspect. The motivation for this approach is very similar to the geometric measure of entanglement first introduced by Shimony (1995) and further extended by Wei and Goldbart (2003). For pure state $|\psi\rangle$, they proposed to adopt the minimal squared distance between $|\psi\rangle$ and the set of separable pure states $|\phi\rangle$ to characterize its entanglement, that is, by minimizing $\min_{|\phi\rangle}\||\psi\rangle - |\phi\rangle\|^2$. Based on this, one can derive the following geometric entanglement measure

$$E_g(\psi) = \min_{|\phi\rangle}(1 - |\langle\psi|\phi\rangle|^2) = 1 - \max_{|\phi\rangle}|\langle\psi|\phi\rangle|^2, \tag{10}$$

where $\psi = |\psi\rangle\langle\psi|$. If ψ is a bipartite state, $E_g(\psi) = 1 - \lambda_{\max}^{1/2}$ (Shimony, 1995), where $\lambda_{\max}^{1/2}$ is the maximal Schmidt coefficient corresponding to the Schmidt decomposition of $|\psi\rangle$ of the following form

$$|\psi\rangle = \sum_i \sqrt{\lambda_i}|\varphi_i^A\rangle \otimes |\varphi_i^B\rangle. \tag{11}$$

For a mixed state described by density operator ρ, the geometric entanglement measure can be defined in terms of the convex roof construction

$$E_g(\rho) = \min_{\{p_i,\psi_i\}}\sum_i p_i E_g(\psi_i), \tag{12}$$

where $\psi_i = |\psi_i\rangle\langle\psi_i|$, and the minimization is with respect to the possible decompositions of

$$\rho = \sum_i p_i\psi_i. \tag{13}$$

Though the calculation of $E_g(\rho)$ for general mixed states is a daunting task, for any two-qubit state ρ, it can be evaluated analytically as

$$E_g(\rho) = \frac{1 - \sqrt{1 - C^2(\rho)}}{2}, \tag{14}$$

where $C(\rho)$ is the concurrence of ρ (Wootters, 1998). We refer to the work of Chen et al. (2014) for a comparison of different geometric entanglement measures.

In the following, we review in detail the geometric measure of discordlike correlations. The motivation for such kind of measures may be fourfold. First, the definition of geometric correlations are based on the idea that *a distance from a given state to the closest state without the desired property is a measure of that property* (Modi et al., 2010), thereby one can quantify amount of correlations by the distance of the considered state to the set of states without the desired property. This endows the resulting geometric measure a clear geometric interpretation. Second, the geometric measures are preferable due to their analytical computability for a wide regime of states. In particular, the theory for geometric entanglement measure is historically well developed, while the features of various distance measures of states are also intensively investigated. The corresponding results can be borrowed for studying geometric discordlike correlations. Thirdly, it is hard to generalize the concept of the entropic discord to multipartite scenario as it is based on QMI which is not defined for multipartite systems. But the geometric approach enables one to define discordlike correlations which are completely applicable for multipartite states. Finally, the geometric discordlike correlations have also been shown to be related to some quantum information processing tasks, thereby endows them with an actual meaning.

Once a distance measure of quantum states is chosen, the corresponding GQD measure will be determined by the set of classical states. In general, the definition of classical states is not unique and different types are studied in different contexts,

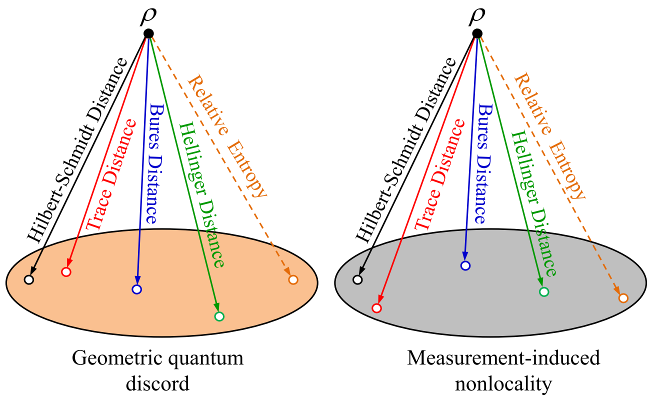

Fig. 1. Geometric quantum correlations of a state ρ can be quantified by the closest (pseudo) distance between it and the set of classical–quantum states (for GQD) and the locally invariant states (for MIN).

see, e.g., the work of Hamieh et al. (2004) and references therein. We refer the following two slightly different types of them which are within the theory of discordlike correlations: partial classical states and total classical states. In the case where there are only two subsystems, they are usually called one-sided (classical–quantum or quantum–classical) and two-sided (classical–classical) classical states. The set contains mixtures of locally distinguishable states and include the set of product states as its subset. They are defined to be classical as the total correlation (measured by QMI) contained in them is the same as the classical correlation (Henderson and Vedral, 2001; Ollivier and Zurek, 2001). In fact, for any partial (total) classical state, there exist local (tensor product of local) POVM such that the postmeasurement state is the same as the premeasurement one. Contrary, if there does not exist such a POVM, the considered state is said to be discordant.

Apart from classical states, we will also consider the sets of locally invariant states and locally invariant projective measurements which are utilized in defining MIN. Here, by saying a projective measurement to be locally invariant we mean that it does not disturb the reduced state (say ρ_A) of a bipartite system AB. It constitutes a subset of the full set of local projective measurements. Moreover, the set of locally invariant states are also different from that of the above-mentioned classical (i.e., zero-discord) states, as some of the classical states may have nonvanishing MIN (Luo and Fu, 2011).

In the following discussion of discordlike correlations other than that measured by relative entropy, we focus our attention mainly on bipartite states. But most of them can be generalized directly to multipartite scenario due to the definite structure of total classical states (Modi et al., 2010).

2.1. Geometric quantum discord

The starting point for the definition of GQD is the identification of the set $\mathcal{CQ}$ of classical–quantum (i.e., zero-discord with respect to subsystem A) states. For a bipartite state in the Hilbert space $\mathcal{H}_{AB}$, the classical–quantum states can be written as

$$\chi = \sum_i p_k \Pi_k^A \otimes \rho_k^B, \tag{15}$$

which is a convex combination of the tensor products of the orthogonal projector Π_k^A in $\mathcal{H}_A$ and an arbitrary density operator ρ_k^B in $\mathcal{H}_B$, with $\{p_k\}$ being any probability distribution. Intuitively, χ of Eq. (15) is said to be classical–quantum as there exists at least one measurement on subsystem A for which B is not affected or in other words, by measuring A one extracts no information about B as the entropy $S(\rho_B)$ and the residual entropy $\sum_k p_k S(\rho_k^B)$ for the conditional ensemble $\{p_k, \rho_k^B\}$ after an optimal local POVM is performed on A are the same. Indeed, within the framework of Henderson and Vedral (2001) and Ollivier and Zurek (2001), one can also check directly that the classical correlation contained in χ is zero.

With $\mathcal{CQ}$ in hand, the category of GQDs for a state ρ can be characterized by its closest (pseudo) distance to the zero-discord state in set $\mathcal{CQ}$ (see Fig. 1). More specifically, it can be formalized in the general form

$$D_{\mathcal{D}}(\rho) = \min_{\chi \in \mathcal{CQ}} \mathcal{D}(\rho, \chi), \tag{16}$$

where $\mathcal{D}(\rho, \chi)$ is a suitable distance measure of quantum states which should satisfy certain natural restrictions in order for the GQD to be well defined, for example, it should be nonnegative, and should be nonincreasing under the action of completely positive and trace preserving (CPTP) map. In certain specific situations, some equivalent forms of $\mathcal{D}(\rho, \chi)$ may be used as well. As the distance between two quantum states can be measured from different aspects, the GQDs can be defined accordingly, provided that they satisfy the conditions for a faithful measure of quantum correlation (Henderson and Vedral, 2001). Moreover, while the GQD defined in Eq. (16) can increase under local operations on party A by its definition, it should not be increased by local operations on the unmeasured party B (Piani, 2012).

Likewise, one could write directly the set $\mathcal{QC}$ of quantum–classical states and define the GQD with respect to subsystem B, or the set $\mathcal{CC}$ of classical–classical states and define the GQD with respect to total system AB. The classical–classical states can be written as $\chi' = \sum_i p_{kl}\Pi_k^A \otimes \Pi_l^B$, and there exists at least one local measurement for which it is not affected. In what follows we consider the GQD defined with respect to A. Its definition with respect to B or AB is similar.

This definition of GQD is somewhat different from the initially proposed entropic measure of QD (Henderson and Vedral, 2001). But it should also satisfy the similar necessary conditions in order for it to be a bona fide measure of quantum correlation, e.g., it is non-negative, vanishes only for zero-discord states, keeps invariant under local unitary transformations, and is nonincreasing under local operations.

2.1.1. Hilbert–Schmidt norm of discord

By using the HS norm as a measure of the distance between two states, Dakić et al. (2010) defined the GQD of ρ as

$$D_G(\rho) = \min_{\chi\in\mathcal{CQ}} \|\rho - \chi\|_2^2, \tag{17}$$

with $\|X\|_2$ denoting the HS norm which is defined as $\|X\|_2 = \sqrt{\mathrm{tr}(X^\dagger X)}$.

Luo and Fu (2010a) further proved that the above definition of GQD is completely equivalent to

$$D_G(\rho) = \min_{\Pi^A} \|\rho - \Pi^A(\rho)\|_2^2, \tag{18}$$

where $\Pi^A = \{\Pi_k^A\}$ is the local von Neumann measurements on party A which sum to the identity (i.e., $\sum_k \Pi_k^A = \mathbb{1}_A$), and

$$\Pi^A(\rho) = \sum_k (\Pi_k^A \otimes \mathbb{1}_B)\rho(\Pi_k^A \otimes \mathbb{1}_B). \tag{19}$$

As the set $\{\Pi^A(\rho)\}$ of postmeasurement states is generally a subset of the full set $\mathcal{CQ}$ of classical states, the equivalence between the above two definitions implies that one only need to take the minimization over $\{\Pi^A(\rho)\}$, and this greatly simplifies the estimation of $D_G(\rho)$. Moreover, Eq. (18) also reveals that $D_G(\rho)$ basically measures how much a measurement on party A does disturb other parts of the state.

Costa and Angelo (2013) put forward another discord measure which was termed as q-discord. It reads

$$D_q(\rho) = \min_{\Pi^A} S_q(\Pi^A[\rho]) - S_q(\rho), \tag{20}$$

where $S_q(\rho)$ is the Tsallis q-entropy defined as (Tsallis, 1988)

$$S_q(\rho) = \frac{1 - \mathrm{tr}\rho^q}{q-1}. \tag{21}$$

It reduces to $-\mathrm{tr}(\rho \ln \rho)$ when $q \to 1$. Moreover, one can obtain immediately from Eq. (20) that $D_2(\rho) = \mathrm{tr}\rho^2 - \mathrm{tr}(\Pi^A[\rho])^2$, thus $D_G(\rho)$ can also be retrieved from the q-discord by setting $q = 2$.

This GQD measure is favored for its ease of computation. In particular, by noting that any two-qubit state ρ can be represented as

$$\rho = \frac{1}{4}\left(\mathbb{1}_4 + \vec{x}\cdot\vec{\sigma}\otimes\mathbb{1}_2 + \mathbb{1}_2\otimes\vec{y}\cdot\vec{\sigma} + \sum_{i,j=1}^{3} r_{ij}\sigma_i\otimes\sigma_j\right), \tag{22}$$

Dakić et al. (2010) derived the explicit formula for $D_G(\rho)$, which is given by

$$D_G(\rho) = \frac{1}{4}\left(\|x\|_2^2 + \|R\|_2^2 - k_{\max}\right), \tag{23}$$

where $\|\vec{x}\|_2^2 = \sum_{i=1}^3 x_i^2$, $\|R\|_2^2 = \mathrm{tr}(R^T R)$, and $k_{\max}$ is the largest eigenvalue of the matrix $K = \vec{x}\vec{x}^T + RR^T$, where $R = (r_{ij})$ is a 3×3 real matrix, and the superscript T denotes transpose of vectors or matrices.

Using the same method, the GQD for any qubit–qutrit state ρ was obtained as (Karpat and Gedik, 2011)

$$D_G(\rho) = \frac{1}{6}\|\vec{x}\|_2^2 + \frac{1}{4}\|R\|_2^2 - k_{\max}, \tag{24}$$

where $\|\vec{x}\|_2^2$, $\|R\|_2^2$, and $k_{\max}$ are similar to those for the two-qubit case, with however $x_i = \mathrm{tr}\rho(\sigma_i\otimes\mathbb{1}_3)$, R becomes a 3×8 matrix with the elements $r_{ij} = \mathrm{tr}\rho(\sigma_i\otimes\lambda_j)$, λ_j $(j = 1, 2, \ldots, 8)$ are the Gell-Mann matrices, and $K = \vec{x}\vec{x}^T/6 + RR^T/4$.

Moreover, for the $(d\times d)$-dimensional Werner state ρ_W and isotropic state ρ_I of the following form

$$\begin{aligned} \rho_W &= \frac{d-x}{d^3-d}\mathbb{1}_{d^2} + \frac{dx-1}{d^3-d}\sum_{ij}|ij\rangle\langle ji|, \quad x\in[-1,1], \\ \rho_I &= \frac{1-x}{d^2-1}\mathbb{1}_{d^2} + \frac{d^2x-1}{d^3-d}\sum_{ij}|ii\rangle\langle jj|, \quad x\in[0,1], \end{aligned} \tag{25}$$

the HS norm of GQD can be obtained analytically as (Luo and Fu, 2010a)

$$D_G(\rho_W) = \frac{(dx-1)^2}{d(d-1)(d+1)^2},$$
$$D_G(\rho_I) = \frac{(d^2x-1)^2}{d(d-1)(d+1)^2}. \tag{26}$$

For any $(m \times n)$-dimensional bipartite state, it can always be decomposed as

$$\rho = \sum_{ij} r_{ij} X_i \otimes Y_j, \tag{27}$$

where $\{X_i : i = 0, 1, \ldots, m^2-1\}$ ($X_0 = \mathbb{1}_m/\sqrt{m}$) is the orthonormal operator basis for subsystem A that satisfy $\mathrm{tr}(X_i^\dagger X_{i'}) = \delta_{ii'}$ (likewise for Y_j), the HS norm of GQD is shown to be lower bounded by (Luo and Fu, 2010a)

$$D_G(\rho) \geq \mathrm{tr}(CC^T) - \sum_{j=1}^{m} \lambda_j = \sum_{j=m+1}^{m^2} \lambda_j, \tag{28}$$

with λ_j representing eigenvalues of the matrix CC^T arranged in nonincreasing order (counting multiplicity), and $C = (r_{ij})$ is a $m^2 \times n^2$ matrix.

Hassan et al. (2012) also obtained a different tight lower bound of $D_G(\rho)$, which is given by

$$D_G(\rho) \geq \|x\|^2 + \|R\|^2 - \sum_{j=1}^{m-1} \eta_j, \tag{29}$$

where η_j are eigenvalues of the matrix $m^2 n(xx^T + RR^T)/2$ arranged in nonincreasing order (counting multiplicity). Here, we have denoted $x = (r_{10}, r_{20}, \ldots, r_{m^2-1,0})^T$, and $R = (r_{kl})$ with $k = 1, 2, \ldots, m^2-1$, and $l = 1, 2, \ldots, n^2-1$. Moreover, note that our decomposed form of ρ in Eq. (27) is slightly different from that given by Hassan et al. (2012), thus induces the seemingly different but essentially the same expressions of the lower bound of $D_G(\rho)$.

Different from the GQD of Eq. (18), Guo et al. (2015) proposed another measure of quantumness by using the average distance between the reduced state $\rho_B = \mathrm{tr}_A \rho$ and the ith output reduced state of subsystem B after the local von Neumann measurements were performed on A. Let $\mathcal{H}_A \otimes \mathcal{H}_B$ with $\dim \mathcal{H}_A = m$ and $\dim \mathcal{H}_B = n \geq m$ be the state space of a bipartite system. The measure is then defined by

$$D_G^{av}(\rho) = \sup_{\Pi^A} \sum_k p_k \|\rho_B - \rho_k^B\|_2^2, \tag{30}$$

where the supremum is taken over the full set of local von Neumann measurements $\Pi^A = \{\Pi_k^A\}$, and $\rho_k^B = \mathrm{tr}_A(\Pi_k^A \otimes \mathbb{1}_B)\rho(\Pi_k^A \otimes \mathbb{1}_B)$. It was shown that only the product states do not contain this kind of quantumness, that is, $D_G^{av}(\rho) = 0$ only for $\rho = \rho_A \otimes \rho_B$. So it captures quantumness of a state which is different from that captured by the QD defined within the framework of Ollivier and Zurek (2001).

While the GQD given in Eq. (17) is analytical computable for any two-qubit state, it is noncontractive, i.e., its value may be changed even by local reversible operations on the unmeasured party B, so it was thought to be not well defined (Piani, 2012). But it does play a role in some quantum information tasks, see Section 2.3. Due to this reason, it is desirable to find ways of characterizing and quantifying GQD using other distance measures of states.

2.1.2. *Trace norm of discord*

In stead of using the HS norm, Paula et al. (2013a) considered the possibility of using the general Schatten p-norm to measure quantum correlations. The Schatten p-norm for a matrix M is defined as

$$\|M\|_p = \{\mathrm{tr}[M^\dagger M]^{p/2}\}^{1/p}, \tag{31}$$

which reduces to the HS norm if $p = 2$, and the trace norm if $p = 1$. By using multiplicative property of the Schatten p-norm under tensor products, Paula et al. (2013a) showed that the corresponding GQD is well defined only for $p = 1$. Based on this fact, they introduced the trace norm of discord as

$$D_T(\rho) = \min_{\chi \in \mathcal{CQ}} \|\rho - \chi\|_1, \tag{32}$$

and for $2 \times n$ dimensional state ρ (i.e., A is a qubit), the optimal χ can also be obtained from the subset $\Pi^A(\rho)$ (Nakano et al., 2013), with $\Pi^A = \{\Pi_k^A\}$ being the set of local projective measurements, i.e.,

$$D_T(\rho) = \min_{\Pi^A} \|\rho - \Pi^A(\rho)\|_1. \tag{33}$$

The calculation of $D_T(\rho)$ is a hard task, and there is no analytical solution for it in general cases. For the two-qubit Bell-diagonal states

$$\rho^{\mathrm{Bell}} = \frac{1}{4}\left(\mathbb{1}_4 + \sum_{i=1}^{3} c_i \sigma_i \otimes \sigma_i\right), \tag{34}$$

it can be derived as

$$D_T(\rho^{\mathrm{Bell}}) = \mathrm{int}\{|c_1|, |c_2|, |c_3|\}, \tag{35}$$

with $\mathrm{int}\{\cdot\}$ denoting the intermediate value. The closest χ_ρ is still a Bell-diagonal state with the only nonzero parameter c_k corresponding to $|c_k| = \max\{|c_1|, |c_2|, |c_3|\}$.

Moreover, for two-qubit X state ρ^X which contains nonzero elements only along the main diagonal and anti-diagonal in the computational basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$, the trace norm of discord is given by (Ciccarello et al., 2014)

$$D_T(\rho^X) = \sqrt{\frac{\xi_1^2 \xi_{\max} - \xi_2^2 \xi_{\min}}{\xi_{\max} - \xi_{\min} + \xi_1^2 - \xi_2^2}}, \tag{36}$$

where

$$\begin{aligned} &\xi_{1,2} = 2(|\rho_{23}| \pm |\rho_{14}|),\ \xi_3 = 1 - 2(\rho_{22} + \rho_{33}), \\ &\xi_{\max} = \max\{\xi_3^2, \xi_2^2 + x_{A3}^2\},\ x_{A3} = 2(\rho_{11} + \rho_{22}) - 1. \end{aligned} \tag{37}$$

For higher-dimensional states, Jakóbczyk et al. (2016) considered a simplified version of $D_T(\rho)$ defined also by Eq. (33), and obtained its analytical solution for certain very special kinds of qutrit–qutrit states, e.g., the maximally entangled states and the Werner states.

The trace norm of discord could also be connected to quantum correlations such as entanglement witness. We refer to the work of Gühne and Tóth (2009) for a comprehensive review of entanglement witnesses. In general, an entanglement witness W is an Hermitian operator for which $\mathrm{tr}(W\rho)$ takes negative value for at least one entangled state and non-negative values for all separable states. By minimizing over the compact subset $\mathcal{M}$ of the set of entanglement witnesses $\mathcal{W}$, one can obtain the optimal entanglement witness, and define the quantifier

$$E_w(\rho) = \max\{0, -\min_{W\in\mathcal{M}} \mathrm{tr}(W\rho)\}, \tag{38}$$

as an entanglement measure (Brandão, 2005).

Debarba et al. (2012) proved that $D_T(\rho)$ is lower bounded by

$$D_T(\rho) \geq \max\{0, -\min_{\{W\in\mathcal{W}|-\mathbb{1}\leq W\leq \mathbb{1}\}} \mathrm{tr}(W\rho)\}. \tag{39}$$

As $E_w(\rho)$ is in fact the negativity $\mathcal{N}(\rho)$ (Vidal and Werner, 2002) for $\mathcal{M} = \{W^{T_A} \in \mathcal{W}|0 \leq W^{T_A} \leq \mathbb{1}\}$ (Brandão, 2005), and the robustness of entanglement $R_r(\rho)/d$ for $\mathcal{M} = \{W \in \mathcal{W}|\mathrm{tr}W = 1\}$ (Vidal and Tarrach, 1999; Brandão and Vianna, 2006), both of which are obviously equal to or smaller than the optimal entanglement witness showed on the right-hand side of Eq. (39), we also have

$$D_T(\rho) \geq \mathcal{N}(\rho),\ \ D_T(\rho) \geq R_r(\rho)/d. \tag{40}$$

While Eq. (32) gives a proper quantum correlation measure, Paula et al. (2013b) further defined the corresponding geometric classical and total correlations using the trace norm. By fixing $\chi \in \Pi^A(\rho)$ and denoting $\tilde{\Pi}^A$ the corresponding optimal measurement operator for obtaining $D_T(\rho)$ (the minimization over different $\Pi^A(\rho)$ is equivalent to the minimization over $\mathcal{CQ}$ for qubit states), they defined the geometric classical correlation $C_T(\rho)$ and total correlation $T_T(\rho)$ as (see Fig. 2)

$$\begin{aligned} &C_T(\rho) = \|\tilde{\Pi}^A(\rho) - \tilde{\Pi}^A(\pi_{\mathrm{rd}})\|_1, \\ &T_T(\rho) = \|\rho - \pi_{\mathrm{rd}}\|_1, \end{aligned} \tag{41}$$

with $\pi_{\mathrm{rd}} = \rho_A \otimes \rho_B$ being product of the reduced density matrices of ρ.

For the Bell-diagonal states ρ^{Bell} of Eq. (34), Paula et al. (2013b) further obtained

$$\begin{aligned} &C_T(\rho^{\mathrm{Bell}}) = c_+, \\ &T_T(\rho^{\mathrm{Bell}}) = \frac{1}{2}[c_+ + \max\{c_+, c_0 + c_-\}], \end{aligned} \tag{42}$$

with c_+, c_-, and c_0 being the maximum, minimum, and intermediate values of $\{|c_1|, |c_2|, |c_3|\}$, respectively. This yields the superadditivity relation: $T_T \leq C_T + D_T$.

In fact, $\tilde{\Pi}^A(\pi_{\mathrm{rd}})$ in Eq. (41) may be not the closest state to $\tilde{\Pi}^A(\rho)$, and π_{rd} composed of the reduced density matrices may also be not the closest product state to ρ. This stimulates more general definitions of geometric classical correlation and total

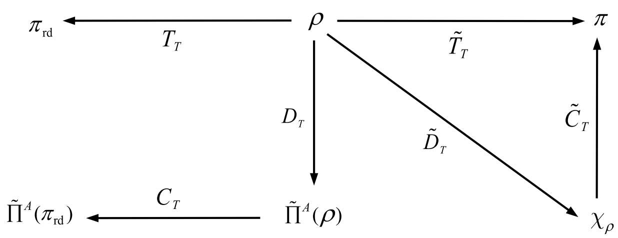

Fig. 2. Geometric correlations in the state ρ, where $\tilde{\Pi}^A$ is the optimal measurement operator for obtaining $D_T(\rho)$, $\pi_{\rm rd} = \rho^A \otimes \rho^B$, χ_ρ is the state with closest trace distance to ρ, and π is the local product state.

correlation. Without loss of generality, one can denote by χ_ρ for the state with closest trace distance to ρ [note that $\tilde{\Pi}^A(\rho)$ is optimal only for A being a qubit], and $\mathcal{P}$ the set of local product states of the subsystems. Based on these, Aaronson et al. (2013b) defined (see Fig. 2)

$$\begin{aligned} \tilde{C}_T(\rho) &= \min_{\pi\in\mathcal{P}} \|\chi_\rho - \pi\|_1, \\ \tilde{T}_T(\rho) &= \min_{\pi\in\mathcal{P}} \|\rho - \pi\|_1, \end{aligned} \tag{43}$$

and derived analytically

$$\tilde{C}_T(\rho^{\rm Bell}) = \sqrt{1+c_+} - 1, \tag{44}$$

where the closest state π_{χ_ρ} to χ_ρ is given by $\pi_{\chi_\rho} = \tilde{\rho}_A \otimes \tilde{\rho}_B$ with $\tilde{\rho}_A = (\mathbb{1}_2 + a_k\sigma_k)/2$ and $\tilde{\rho}_B = (\mathbb{1}_2 + b_k\sigma_k)/2$. The index k corresponds to the maximum of $|c_k| = c_+$, and $a_k = b_k|c_k|/c_k$. Clearly, π_{χ_ρ} is not the product of its marginals, and is not even a Bell-diagonal state in general.

For the family of classical–quantum or quantum–classical bipartite states, Ma et al. (2014) introduced another quantum correlation measure based on the non-commutativity of quantum observables, where the trace norm of the commutators of the ensemble state of one subsystem is used. To be explicit, for classical–quantum state χ of Eq. (15) described by the ensemble $\{X_i\}$ with $X_i = p_i\rho_i$, they found the quantity

$$D(\chi) = \sum_{i>j} \|[X_i, X_j]\|_1, \tag{45}$$

satisfies the following properties of correlations: $D(\chi) \geq 0$, and the equality holds when subsystem B is also classical. Moreover, it is local unitary invariant, and is nonincreasing when an ancillary system is introduced.

The above result was further extended by Guo (2016), who proposed to define GQD for any ρ in a similar manner. Let $\{|i^A\rangle\}$ be an orthonormal basis of $\mathcal{H}_A$. Then any state ρ acting on $\mathcal{H}_A \otimes \mathcal{H}_B$ can be represented by

$$\rho = \sum_{i,j} |i^A\rangle\langle j^A| \otimes B_{ij}. \tag{46}$$

with B_{ij} being operators in $\mathcal{H}_B$. The non-commutativity measure of GQD for ρ is defined by

$$D_N(\rho) := \frac{1}{2} \sum_{(ij)\neq(kl)} \|[B_{ij}, B_{kl}]\|_1, \tag{47}$$

under the trace norm, and

$$D_N'(\rho) := \frac{1}{2} \sum_{(ij)\neq(kl)} \|[B_{ij}, B_{kl}]\|_2, \tag{48}$$

under the HS norm, where the commutator $[X, Y] = XY - YX$, and the summation is over all different pairs of $\{B_{ij}\}$.

These two measures of GQD can be calculated easily for ρ of arbitrary dimension. In particular, for pure state $\psi = |\psi\rangle\langle\psi|$ with Schmidt decomposition of Eq. (11), analytical solutions of them are given by

$$\begin{aligned} D_N(\psi) &= 2\sum_{i,j} \lambda_i\lambda_j \left(\sum_{(k,l)\in\Omega} \lambda_k\lambda_l \right), \\ D_N'(\psi) &= 2\sum_{i,j} \lambda_i\lambda_j \left(\sum_{(k,l)\in\Omega'} \lambda_k\lambda_l \right) + \sqrt{2}, \end{aligned} \tag{49}$$

Please cite this article in press as: Hu M.-L., et al., Quantum coherence and geometric quantum discord. Physics Reports (2018), https://doi.org/10.1016/j.physrep.2018.07.004.

where $\Omega = \{(k, l)\}$ with $i < k \le j \le l$ or $k = i, l = j$ if $i < j$, and $i \le k < l$ if $i = j$, while $\Omega' = \{(k, l)\}$ with $i < k \le j \le l$ if $i < j$, and $i \le k < l$ if $i = j$.

It was shown via several examples that they can reflect the amount of the original QD. In particular, these two measures disappear if and only if the corresponding state is zero discordant. Here, we would like to further point out that this is in fact a direct consequence of the result of Chen et al. (2011), in which a necessary and sufficient condition for vanishing QD has been proven. It says that ρ has zero QD if and only if all the operators $\rho_{B|ij}$ commute with each other for any orthonormal basis $\{|i^A\rangle\}$ in $\mathcal{H}_A$, where

$$\rho_{B|ij} := \langle i^A|\rho|j^A\rangle. \tag{50}$$

It is obvious that B_{ij} in Eq. (46) is the same as $\rho_{B|ij}$ of the above equation.

2.1.3. Bures distance of discord

The distance between two states ρ and σ can also be quantified by the Bures distance

$$\mathcal{D}_B(\rho, \sigma) = 2[1 - \sqrt{F(\rho, \sigma)}], \tag{51}$$

where

$$F(\rho, \sigma) = \left[\mathrm{tr}(\sqrt{\rho}\sigma\sqrt{\rho})^{1/2}\right]^2, \tag{52}$$

is the Uhlmann fidelity (Nielsen and Chuang, 2000). The Bures distance satisfies the preferable properties of joint convexity, i.e.,

$$\begin{aligned}\mathcal{D}_B(p_1\rho_1 + p_2\rho_2, p_1\sigma_1 + p_2\sigma_2) \le & p_1\mathcal{D}_B(\rho_1, \sigma_1) \\ & + p_2\mathcal{D}_B(\rho_2, \sigma_2),\end{aligned} \tag{53}$$

and it is also monotonous under CPTP maps. It has been used to quantify entanglement (Vedral and Plenio, 1998; Streltsov et al., 2010b), and there are some equivalent definitions of Bures distance discord. Spehner and Orszag (2013) proposed to define it as

$$D_B(\rho) = (2 + \sqrt{2})\left[1 - \sqrt{F_{\max}(\rho)}\right], \tag{54}$$

where $F_{\max}(\rho) = \max_{\chi\in\mathcal{C}_Q} F(\rho, \chi)$ represents the maximum of the Uhlmann fidelity, and the constant $2 + \sqrt{2}$ is introduced for the normalization of it for two-qubit maximally discordant states. Moreover, the square root of $D_B(\rho)$ in Eq. (54) equals to that defined by Aaronson et al. (2013a).

There are several cases that the evaluation of $F_{\max}(\rho)$, and thus $D_B(\rho)$ can be simplified:

(1) For arbitrary pure state $|\psi\rangle$, the maximum Uhlmann fidelity can be obtained as $F_{\max}(|\psi\rangle\langle\psi|) = \mu_{\max}$, with $\mu_{\max}$ being the largest Schmidt coefficient of $|\psi\rangle$ (Spehner and Orszag, 2013).

(2) For any Bell-diagonal state ρ^{Bell} of Eq. (34), we have (Aaronson et al., 2013a; Spehner and Orszag, 2014)

$$\begin{aligned}F_{\max}(\rho^{\mathrm{Bell}}) = \frac{1}{2} + \frac{1}{4}\max_{\langle ijk\rangle}\Bigg[&\sqrt{(1 + c_i)^2 - (c_j - c_k)^2} \\ & + \sqrt{(1 - c_i)^2 - (c_j + c_k)^2}\Bigg],\end{aligned} \tag{55}$$

where the maximum is taken over all the cyclic permutations of $\{1, 2, 3\}$.

(3) For general $(2 \times n)$-dimensional state, although there is no analytical solution, the maximum of the Uhlmann fidelity can be calculated as (Spehner and Orszag, 2014)

$$F_{\max}(\rho) = \frac{1}{2}\max_{\|\vec{u}=1\|}\left(1 - \mathrm{tr}\Lambda(\vec{u}) + 2\sum_{k=1}^{n_B}\lambda_k(\vec{u})\right), \tag{56}$$

where $\lambda_k(\vec{u})$ are eigenvalues of

$$\Lambda(\vec{u}) = \sqrt{\rho}(\sigma_{\vec{u}} \otimes \mathbb{1}_B)\sqrt{\rho} \tag{57}$$

arranged in nonincreasing order, and $\sigma_{\vec{u}} = \vec{u} \cdot \vec{\sigma}$, with $\vec{u} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$ being a unit vector in $\mathbb{R}^3$, and n_B the dimension of $\mathcal{H}_B$.

2.1.4. Relative entropy of discord

The relative entropy of a state ρ to another state σ is defined as

$$S(\rho \parallel \sigma) = \mathrm{tr}(\rho\log_2\rho) - \mathrm{tr}(\rho\log_2\sigma), \tag{58}$$

which is non-negative, and can sometimes be infinite. Though technically the relative entropy does not have a geometric interpretation as $S(\rho \parallel \sigma) \ne S(\sigma \parallel \rho)$ in general, it can be recognized as a (pseudo) distance measure of quantum states.

The relative entropy has been used to define quantum entanglement,

$$E_R = \min_{\sigma \in \mathcal{S}} S(\rho \parallel \sigma), \tag{59}$$

which is indeed the minimal relative entropy of ρ to the set $\mathcal{S}$ of separable states (Vedral et al., 1997; Vedral and Plenio, 1998). In the same spirit, one can use it to define the discordlike correlation measures. Modi et al. (2010) made the first attempt in this direction by introducing the relative entropy of discord D_R and the relative entropy of dissonance Q_R, They are defined, respectively, to be the minimal relative entropy of ρ and σ (the closest separable state to ρ) to the set of classical states $\mathcal{C}$ (here, by saying a state to be classical, we mean that it is classical with respect to all of its subsystems, which is similar to the set of $\mathcal{CC}$ states for bipartite systems), and can be written explicitly as

$$D_R = \min_{\chi \in \mathcal{C}} S(\rho \| \chi), \quad Q_R = \min_{\chi \in \mathcal{C}} S(\sigma \| \chi), \tag{60}$$

which are applicable for the general bipartite and multipartite states. In particular, $Q_R(\sigma)$ reveals a kind of quantum correlation excluding quantum entanglement.

Modi et al. (2010) also showed that D_R and Q_R are equivalent to

$$D_R = S(\chi_\rho) - S(\rho), \quad Q_R = S(\chi_\sigma) - S(\rho), \tag{61}$$

where $S(\chi_\rho) = \min_{|\vec{k}\rangle} S(\sum_{\vec{k}} |\vec{k}\rangle\langle\vec{k}|\rho|\vec{k}\rangle\langle\vec{k}|)$ with $\{|\vec{k}\rangle\}$ forming the eigenbasis of χ_ρ, and likewise for $S(\chi_\sigma)$. So the optimization in Eq. (59) is reduced to the optimization of the von Neumann entropy $S(\chi_\rho)$ and $S(\chi_\sigma)$.

In a similar manner to Eqs. (60) and (61), Modi et al. (2010) defined the total correlation and classical correlation as

$$\begin{aligned} T_\rho &= S(\rho \parallel \pi_\rho) = S(\pi_\rho) - S(\rho), \\ T_\sigma &= S(\sigma \parallel \pi_\sigma) = S(\pi_\sigma) - S(\sigma), \\ C_\rho &= S(\chi_\rho \parallel \pi_{\chi_\rho}) = S(\pi_{\chi_\rho}) - S(\chi_\rho), \\ C_\sigma &= S(\chi_\sigma \parallel \pi_{\chi_\sigma}) = S(\pi_{\chi_\sigma}) - S(\chi_\sigma), \end{aligned} \tag{62}$$

where $\pi_\rho = \pi_1 \otimes \cdots \otimes \pi_N$ (π_k is the reduced density operator of the kth subsystem of ρ), and likewise for π_σ, π_{χ_ρ}, and π_{χ_σ}. From these definitions, one can obtain the following additivity relations

$$T_\rho + L_\rho = D_R + C_\rho, \quad T_\sigma + L_\sigma = Q_R + C_\sigma, \tag{63}$$

where $L_\rho = S(\pi_{\chi_\rho}) - S(\pi_\rho)$ and $L_\sigma = S(\pi_{\chi_\sigma}) - S(\pi_\sigma)$.

For two-qubit Bell-diagonal states of Eq. (34), if we rewrite it as $\rho^{\text{Bell}} = \sum_i \lambda_i |\Psi_i\rangle\langle\Psi_i|$, where λ_i are arranged in nonincreasing order and $|\Psi_i\rangle$ ($i = 1, 2, 3, 4$) are the four Bell states, then the closest separable state to it is given by (Vedral and Plenio, 1998)

$$\sigma = \sum_{i=1}^{4} p_i |\Psi_i\rangle\langle\Psi_i|, \tag{64}$$

where $p_1 = 1/2$ and $p_i = \lambda_i/[2(1 - \lambda_1)]$ for $i \neq 1$. Similarly, the closest classical state to ρ^{Bell} is given by (Modi et al., 2010)

$$\begin{aligned} \chi_\rho = &\frac{q_\rho}{2}[|\Psi_1\rangle\langle\Psi_1| + |\Psi_2\rangle\langle\Psi_2|] \\ &+ \frac{1 - q_\rho}{2}[|\Psi_3\rangle\langle\Psi_3| + |\Psi_4\rangle\langle\Psi_4|], \end{aligned} \tag{65}$$

with $q_\rho = \lambda_1 + \lambda_2$, and the closest classical state to σ can be obtained directly by substituting q_ρ with $q_\sigma = p_1 + p_2$.

Moreover, it is worthwhile to note that for any bipartite state ρ_{AB}, the relative entropy of discord $D_R(\rho_{AB})$ equals the zero-way quantum deficit

$$\Delta^{\emptyset}(\rho_{AB}) = \min_{\Pi^A \otimes \Pi^B} S(\rho_{AB} \parallel \Pi^A \otimes \Pi^B[\rho_{AB}]), \tag{66}$$

which is also a discordlike quantum correlation measure and was defined originally from the perspective of work extraction from the quantum system coupled to a heat bath (Horodecki et al., 2005a). The above equation thereby endows $\Delta^{\emptyset}(\rho_{AB})$ a geometric interpretation, that is, it corresponds to the minimal relative entropy between ρ_{AB} and the full set of postmeasurement states $\Pi^A \otimes \Pi^B[\rho_{AB}]$.

The one-way quantum deficit can also be expressed by using the quantum relative entropy as (Horodecki et al., 2005a)

$$\Delta^{\rightarrow}(\rho_{AB}) = \min_{\Pi^A} S(\rho_{AB} \parallel \Pi^A[\rho_{AB}]), \tag{67}$$

and it also equals the minimal relative entropy between ρ_{AB} and the set $\mathcal{CQ}$ of classical–quantum states. Furthermore, $\Delta^{\rightarrow}(\rho_{AB})$ also equivalents to the thermal QD $\tilde{D}_A(\rho_{AB})$ introduced by Zurek (2003).

2.1.5. Hellinger distance of discord

Although in most cases the quantum correlation measure is defined as a direct function of the density operator ρ itself, its other forms may also be very useful. For example, with roots in the well-known notion of WY skew information (Wigner and Yanase, 1963), the square root $\sqrt{\rho}$ has been used to study the local quantum uncertainty (LQU) of a single system (Girolami et al., 2013).

By using the square root form of a density operator, Chang and Luo (2013) introduced a new quantifier of the GQD, for which we call it the Hellinger distance discord. It can be recognized as a modified version of the GQD proposed by Dakić et al. (2010), and reads

$$D_H(\rho) = 2\min_{\Pi^A} \|\sqrt{\rho} - \Pi^A(\sqrt{\rho})\|_2^2, \tag{68}$$

where the minimum is taken over $\Pi^A = \{\Pi_k^A\}$, with

$$\Pi^A(\sqrt{\rho}) = \sum_k (\Pi_k^A \otimes \mathbb{1}_B)\sqrt{\rho}(\Pi_k^A \otimes \mathbb{1}_B), \tag{69}$$

and $\mathbb{1}_B$ is the identity operator in $\mathcal{H}_B$.

The Hellinger distance discord is well defined. It is locally unitary invariant, and vanishes if and only if ρ is a classical–quantum state. It also keeps invariant when adding a local ancilla to the unmeasured party, i.e., $D_H(\rho_{A:BC}) = D_H(\rho_{A:B})$ for $\rho_{A:BC} = \rho_{AB} \otimes \rho_C$. This property averts the fault encountered when measuring GQD via the HS norm (Piani, 2012). Moreover, it is similar to the squared form of the Hellinger distance defined as

$$d_H^2(\rho, \chi) = \frac{1}{2}\mathrm{tr}\{(\sqrt{\rho} - \sqrt{\chi})^2\}, \tag{70}$$

and this is the reason for it to be called the Hellinger distance discord.

For pure state $\psi = |\psi\rangle\langle\psi|$ with the Schmidt decomposition of Eq. (11), the Hellinger distance discord can be obtained as $D_H(\psi) = 1 - \sum_i \lambda_i^2$, which is the same as that of the GQD based on the HS norm (Dakić et al., 2010). Moreover, for the Bell-diagonal state ρ^{Bell}, it is given by

$$D_H(\rho^{\mathrm{Bell}}) = 1 - \frac{1}{4}(h^2 + \max_i\{d_i^2\}), \tag{71}$$

where $h = \sum_i \sqrt{\lambda_i}$, $d_i = h - 2\sqrt{\lambda_4} - 2\sqrt{\lambda_i}$ $(i = 1, 2, 3)$, and λ_i are eigenvalues of ρ^{Bell} given by

$$\begin{aligned} \lambda_1 &= \frac{1}{4}(1 - c_1 + c_2 + c_3),\ \lambda_2 = \frac{1}{4}(1 + c_1 - c_2 + c_3), \\ \lambda_3 &= \frac{1}{4}(1 + c_1 + c_2 - c_3),\ \lambda_4 = \frac{1}{4}(1 - c_1 - c_2 - c_3). \end{aligned} \tag{72}$$

For $(2 \times n)$-dimensional state ρ with the decomposed form of

$$\sqrt{\rho} = \sum_{ij} \gamma_{ij} X_i \otimes Y_j, \tag{73}$$

where $\{X_i : i = 0, 1, 2, 3\}$ and $\{Y_j : j = 0, 1, \ldots, n^2 - 1\}$ constitutes the orthonormal operator bases for the Hilbert spaces $\mathcal{H}_A$ and $\mathcal{H}_B$, the Hellinger distance discord can be calculated as (Chang and Luo, 2013)

$$D_H(\rho) = 2(1 - \|\mathbf{r}\|_2^2 - \mu_{\max}), \tag{74}$$

where $\|\mathbf{r}\|_2^2 = \sum_j \gamma_{0j}^2$, and $\mu_{\max}$ represents the largest eigenvalue of the matrix $\Gamma\Gamma^\dagger$, with $\Gamma = (\gamma_{ij})_{i=1,2,3;j=0,1,\ldots,n^2-1}$.

2.1.6. Local quantum uncertainty

The WY skew information was defined as follows (Wigner and Yanase, 1963)

$$\mathrm{I}^p(\rho, K) = -\frac{1}{2}\mathrm{tr}\{[\rho^p, K][\rho^{1-p}, K]\}, \tag{75}$$

with $p \in (0, 1)$, and when $p = 1/2$ (we omit the superscript in $\mathrm{I}^p(\rho, K)$ for brevity),

$$\mathrm{I}(\rho, K) = -\frac{1}{2}\mathrm{tr}\{[\sqrt{\rho}, K]^2\} = \frac{1}{2}\|[K, \sqrt{\rho}]\|_2^2, \tag{76}$$

was also termed the WY skew information, where K denotes the observable to be measured (a self-adjoint operator). $\mathrm{I}(\rho, K)$ measures the information content embodied in a state that is skewed to the chosen observable K, and is bounded above by the variance of K, i.e.,

$$\mathrm{I}(\rho, K) \leq \langle K^2\rangle_\rho - \langle K\rangle_\rho^2, \tag{77}$$

where the equality holds for pure states. This equation shows that $\mathrm{I}(\rho, K)$ is indeed a lower bound of the weighted statistical uncertainty about K (measured by the variance of K) for any possible state preparation. For $\rho = \sum_i \lambda_i |\psi_i\rangle\langle\psi_i|$, one can further obtain

$$\mathrm{I}(\rho, K) = \frac{1}{2}\sum_{ij}(\sqrt{\lambda_i} - \sqrt{\lambda_j})^2 K_{ij}^2, \tag{78}$$

where the overlap $K_{ij} = |\langle\psi_i|K|\psi_j\rangle|$.

Compared to the variance of K, the skew information has many advantages. In particular, it possesses preferable properties which are useful for defining quantum correlations, e.g., it is nonnegative and vanishes if and only if $[\rho, K] = 0$, it is convex, that is, $\mathrm{I}(\sum_i p_i\rho_i, K) \le \sum_i p_i \mathrm{I}(\rho_i, K)$. Indeed, the skew information can be used to characterize uncertainty relation (Luo, 2003), while a correlation measure based on it has also been introduced (Luo et al., 2012).

Girolami et al. (2013) proposed to use the WY skew information to quantify LQU of a bipartite state ρ. They chose the local observable $K^\Lambda = K_A^\Lambda \otimes \mathbb{1}_B$ (Λ denotes spectrum of K_A^Λ that are nondegenerate as this corresponds to maximally informative observables on A) and defined the LQU as

$$\mathcal{U}_A^\Lambda(\rho) = \min_{K^\Lambda} \mathrm{I}(\rho, K^\Lambda), \tag{79}$$

which is not only a measure of uncertainty, but also a well-defined quantum correlation measure. In particular, for $(2\times n)$-dimensional state ρ, by dropping the superscript Λ for brevity and choosing the nondegenerate observables as $K_A = \vec{n}\cdot\vec{\sigma}^A$, with $\vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ being the vector of Pauli operators, the LQU can be derived as (Girolami et al., 2013)

$$\mathcal{U}_A(\rho) = 1 - \lambda_{\max}(W_{AB}), \tag{80}$$

where $\lambda_{\max}(W_{AB})$ denotes the maximum eigenvalue of the matrix W_{AB} whose elements are given by

$$(W_{AB})_{ij} = \mathrm{tr}\{\sqrt{\rho}(\sigma_i^A \otimes \mathbb{1}_B)\sqrt{\rho}(\sigma_j^B \otimes \mathbb{1}_B)\}, \tag{81}$$

from which one can obtain that for pure state $|\psi\rangle$, the LQU reduces to the linear entropy of entanglement

$$\mathcal{U}_A(|\psi\rangle\langle\psi|) = 2[1 - \mathrm{tr}(\rho_A)^2], \tag{82}$$

where $\rho_A = \mathrm{tr}_B(|\psi\rangle\langle\psi|)$.

In fact, for arbitrary $(2\times n)$-dimensional state ρ, as $K_A = \vec{n}\cdot\vec{\sigma}^A$ is a root-of-unity local unitary operation, which implies $K_A f(\rho) K_A = f(K_A\rho K_A)$ for arbitrary function $f(\cdot)$, thus

$$\begin{aligned}\mathcal{U}_A(\rho) &= 1 - \mathrm{tr}\{\sqrt{\rho}K_A\sqrt{\rho}K_A\},\\ &= 1 - \mathrm{tr}\{\sqrt{\rho}\sqrt{K_A\rho K_A}\},\\ &= d_H^2(\rho, K_A\rho K_A),\end{aligned} \tag{83}$$

while $\{\Pi_k^A\}$ of Eq. (69) can be written as $\Pi_{1,2}^A = (\mathbb{1}_2 \pm K_A)/2$, which gives

$$\begin{aligned}\left[\sqrt{\rho} - \Pi^A(\sqrt{\rho})\right]^2 =& \frac{1}{4}(\rho + K_A\rho K_A - \sqrt{\rho}K_A\sqrt{\rho}K_A\\ &- K_A\sqrt{\rho}K_A\sqrt{\rho}),\end{aligned} \tag{84}$$

then by combining this with Eqs. (68) and (79), and (83), one can obtain

$$\mathcal{U}_A(\rho) = 2D_H(\rho). \tag{85}$$

This equation establishes a direct connection between LQU and the Hellinger distance discord, thereby gives LQU a geometric interpretation, although it applies only for $(2\times n)$-dimensional states.

By restricting K_A to rank-one projectors, Yu et al. (2014) further defined a measure of quantum correlation for arbitrary bipartite state as follows

$$Q_A(\rho) = \min_{K_A}\sum_{i=1}^m \mathrm{I}(\rho, K_A^i \otimes \mathbb{1}_B), \tag{86}$$

where the minimization is taken over the set of $K_A = \{|i^A\rangle\langle i^A|\}$, and m is the dimension of $\mathcal{H}_A$. The measure $Q_A(\rho)$ vanishes if and only if ρ is classical–quantum correlated (i.e., $\rho \in \mathcal{CQ}$). Moreover, it is locally unitary invariant, and is contractive under CPTP map on the unmeasured party B.

Yu et al. (2014) also gave a numerical method for calculating $Q_A(\rho)$ which uses the technique of approximate joint diagonalization. Moreover, for any pure state $|\psi\rangle$ and $(2\times n)$-dimensional state ρ, analytical solutions of $Q_A(\rho)$ can be obtained, which equal half of $\mathcal{U}_A(\rho)$. For the $(d\times d)$-dimensional Werner state ρ_W and isotropic state ρ_I of the form of Eq. (25), one has

$$\begin{aligned} Q_A(\rho_W) &= \frac{d-x-\sqrt{(d^2-1)(1-x^2)}}{2(d+1)}, \\ Q_A(\rho_I) &= \frac{1-2\sqrt{(d^2-1)(1-x)x}+(d^2-2)x}{d(d+1))}. \end{aligned} \tag{87}$$

2.1.7. Negativity of quantumness

The quantumness in a bipartite or multipartite state can also be quantified by virtue of the amount of entanglement created between the considered system and the measurement apparatus in a local measurement. Streltsov et al. (2011b) made such an attempt along this line. For a bipartite state ρ_{AB} and a measurement apparatus M prepared in an initial state $|0^M\rangle$, they proved that the created minimum distillable entanglement between M and AB equals the one-way deficit $\Delta^{\rightarrow}(\rho^{AB})$ [see Eq. (67)], that is,

$$\Delta^{\rightarrow}(\rho^{AB}) = \min_{U_{MAB}} E_D^{M:AB}(U_{MAB}\rho_{MAB}U_{MAB}^{\dagger}), \tag{88}$$

where $\rho_{MAB}=|0^M\rangle\langle 0^M|\otimes\rho_{AB}$, $U_{MAB}=U_{MA}\otimes\mathbb{1}_B$ and U_{MA} denotes only those unitary operators which give $\sum_k \Pi_k^A\rho_{AB}\Pi_k^A = \mathrm{tr}_M(U\rho_{MAB}U^{\dagger})$. Therefore, the above equation establishes a quantitative connection between discordlike quantum correlation and entanglement.

Nakano et al. (2013) further proposed several discordlike measures of quantumness by using this approach. First, they introduced the measurement interaction $V_{A\mapsto AA'}$ described by a linear isometry from A to a bipartite system AA', i.e.,

$$V_{A\mapsto AA'}|a_i\rangle = |a_i\rangle|i'\rangle,\ \forall i, \tag{89}$$

with $\{|a_i\rangle\}$ being the basis of system A, and $\{|i'\rangle\}$ is the computational basis of system A'. Then, for any N-partite system described by density operator $\rho_{\boldsymbol{A}}$ (we denote $\boldsymbol{A}=A_1A_2\ldots A_N$ for short) and the chosen subsystems $\Sigma\subseteq\{A_1A_2\ldots A_N\}$ for which the measurements are performed, the corresponding premeasurement state reads

$$\tilde{\rho}_{\Xi} = \left(\bigotimes_{i\in\Sigma} V_{i\mapsto ii'}\right)\rho_{\boldsymbol{A}}\left(\bigotimes_{i\in\Sigma} V_{i\mapsto ii'}\right)^{\dagger}, \tag{90}$$

where $\Xi=\boldsymbol{A}\cup\Sigma'$.

By using negativity as a measure of entanglement (Vidal and Werner, 2002), Nakano et al. (2013) defined negativity of quantumness as the minimum entanglement created between the system and the apparatus, that is,

$$Q_{\mathcal{N}}^{\Sigma}(\rho_{\boldsymbol{A}}) := \min_{\tilde{\rho}^{\Xi}} \mathcal{N}_{\boldsymbol{A}:\Sigma'}(\tilde{\rho}_{\Xi}), \tag{91}$$

where the minimization is taken over all possible $\tilde{\rho}_{\Xi}$ obtained with different choice of basis for the system $\boldsymbol{A}$. This measure is shown to be nonnegative for any $\rho_{\boldsymbol{A}}$, and vanishes if and only if $\rho_{\boldsymbol{A}}$ is classical on the subsystems Σ to be measured. When $\Sigma=\{k_1,k_2,\ldots,k_n\}$ $(n<N)$, $Q_{\mathcal{N}}^{\Sigma}(\rho_{\boldsymbol{A}})$ is said to be the partial negativity of quantumness and is equivalent to (Nakano et al., 2013)

$$Q_{\mathcal{N}}^{\Sigma}(\rho_{\boldsymbol{A}}) = \min_{\bigotimes_{k\in\Sigma}\mathcal{B}_k} \frac{1}{2}\left(\sum_{i_{k_1},i_{k_2},\ldots,i_{k_n}} \|\rho_{i_{k_1},i_{k_2},\ldots,i_{k_n}}\|_1 - 1\right), \tag{92}$$

where $\mathcal{B}_k=\{|a_{i_k}^{(k)}\rangle\}$ denotes basis of the kth subsystem, and

$$\rho_{i_{k_1},i_{k_2},\ldots,i_{k_n}} = \langle a_{i_{k_1}}^{(k_1)}a_{i_{k_2}}^{(k_2)}\ldots a_{i_{k_n}}^{(k_n)}|\rho_{\boldsymbol{A}}|a_{i_{k_1}}^{(k_1)}a_{i_{k_2}}^{(k_2)}\ldots a_{i_{k_n}}^{(k_n)}\rangle. \tag{93}$$

When $\Sigma=\boldsymbol{A}$, one obtains the total negativity of quantumness, and it is equivalent to (Nakano et al., 2013)

$$Q_{\mathcal{N}}^{\boldsymbol{A}}(\rho^{\boldsymbol{A}}) = \min_{\bigotimes_{k=1}^N\mathcal{B}_k} \frac{1}{2}\Big(\|\rho^{\boldsymbol{A}}\|_{l_1} - 1\Big), \tag{94}$$

where the minimization is taken over different choices of factorized basis $\bigotimes_{k=1}^N\mathcal{B}_k$ and the l_1 norm is also calculated in the same basis.

For the case of bipartite state ρ_{AB} with $\dim A=2$ (i.e., A is a qubit), Nakano et al. (2013) further showed that

$$\begin{aligned} Q_{\mathcal{N}}^{A}(\rho_{AB}) &= \frac{1}{2}\min_{\Pi^A}\|\rho_{AB}-\Pi^A(\rho_{AB})\|_1, \\ &= \frac{1}{2}\min_{\sigma\in\mathcal{CQ}}\|\rho_{AB}-\sigma\|_1, \end{aligned} \tag{95}$$

where $\Pi^A = \{\Pi_i^A\}$ with $\Pi_i^A = |a_i\rangle\langle a_i|$ being the local projective measurements on A. It implies that the minimization over the full set of classical–quantum states can be simplified to the minimization only over the full set of postmeasurement states. Similarly, the total negativity of quantumness for the above-mentioned ρ_{AB} is given by

$$\begin{aligned} Q_{\mathcal{N}}^{AB}(\rho_{AB}) &= \frac{1}{2}\min_{\Pi^A\otimes\Pi^B}\|\rho_{AB}-\Pi^A\otimes\Pi^B(\rho_{AB})\|_{l_1}, \\ &= \frac{1}{2}\min_{\sigma\in\mathcal{CC}}\|\rho_{AB}-\sigma\|_{l_1} \end{aligned} \tag{96}$$

where the local projective measurements $\Pi^A\otimes\Pi^B$ are defined with respect to the factorized basis $\mathcal{B}_A\otimes\mathcal{B}_B$, and the l_1 norm in the first line is also calculated with the same basis, while that in the second line is calculated with respect to the eigenbasis of σ (if the eigenbasis is degenerate then it is chosen optimally to minimize the distance by default).

For certain special ρ_{AB}, analytical solutions of $Q_{\mathcal{N}}^{A}(\rho_{AB})$ and $Q_{\mathcal{N}}^{AB}(\rho_{AB})$ can be obtained. For example, for the two-qubit ρ_{AB} with $\rho_A = \mathbb{1}_2/2$, one has $Q_{\mathcal{N}}^{A}(\rho_{AB}) = \mathrm{int}\{s_1, s_2, s_3\}/2$, where s_i is the singular value of $R = (r_{ij})$ with $r_{ij} = \mathrm{tr}(\rho_{AB}\sigma_i\otimes\sigma_j)$. If ρ_{AB} belongs to the Bell-diagonal states, one can further obtain $Q_{\mathcal{N}}^{AB}(\rho_{AB}) = \mathrm{int}\{s_1, s_2, s_3\}/2$. Moreover, for Werner states and isotropic states given in Eq. (25), one has (Nakano et al., 2013)

$$\begin{aligned} Q_{\mathcal{N}}^{A}(\rho_W) = Q_{\mathcal{N}}^{AB}(\rho_W) &= \frac{|dx-1|}{2(d+1)}, \\ Q_{\mathcal{N}}^{A}(\rho_I) = Q_{\mathcal{N}}^{AB}(\rho_I) &= \frac{|d^2x-1|}{d+1}. \end{aligned} \tag{97}$$

2.2. Measurement-induced nonlocality

Apart from the various Bell-type nonlocality widely studied in the literature (Genovese, 2005), the nonlocality of a system can also be studied from other aspects. One typical research direction in recent years is initialized by Luo and Fu (2011), who proposed the notion of MIN. In this subsection, we will review in detail various geometric measures of them. They were all defined from the measurement perspective, and were motivated by those of the discordlike correlation measures (Modi et al., 2012). We shall focus mainly on the bipartite systems described by the density operator ρ in the Hilbert space $\mathcal{H}_A\otimes\mathcal{H}_B$. But the related concepts and ideas can in fact be generalized to multipartite systems straightforwardly.

Different from the definitions of GQDs in the above section, and motivated by the consideration that the state of a bipartite system may be disturbed by a measurement on one party (say A) of the considered system, one can define the MIN as the maximal distance that a state ρ to the set $\mathcal{L}$ of locally invariant quantum states, namely

$$N(\rho) = \max_{\delta\in\mathcal{L}} \mathcal{D}(\rho,\delta), \tag{98}$$

where the locally invariant of δ means that $\delta = \sum_k \Pi_k^A\rho\Pi_k^A$ for all $\Pi^A = \{\Pi_k^A\}$ satisfying $\sum_k \Pi_k^A\rho_A\Pi_k^A = \rho_A$.

By adopting different distance measures, one can define different measures of MIN which possess distinct novel characteristics. Moreover, for bipartite state ρ with nondegenerate reduced state ρ_A, the MIN measures can readily be obtained as the optimal measurements $\tilde{\Pi}^A = \{\tilde{\Pi}_i^A\}$ are induced by the spectral resolutions of $\rho_A = \sum_i p_i^A\tilde{\Pi}_i^A$. But when ρ_A is degenerate, an optimization procedure should be performed. In fact, seeking the optimal measurements in order to extract various measurement-based correlations (including MIN) is an important task for characterizing quantumness of a state (Hamieh et al., 2004; Amico et al., 2012).

2.2.1. Hilbert–Schmidt norm of MIN

The notion of MIN was introduced by Luo and Fu (2011). They used the HS norm as a measure of distance, and defined the MIN as

$$N_G(\rho) = \max_{\Pi^A}\|\rho-\Pi^A(\rho)\|_2^2, \tag{99}$$

with Π^A being the locally invariant projective measurements. $N_G(\rho)$ characterizes the maximal global disturbance caused by the locally invariant measurements, in the sense that it corresponds to the maximal square HS distance between the postmeasurement state $\Pi^A(\rho)$ and the premeasurement state ρ. The way for revealing nonlocal feature of a state ρ by doing local measurements on one of its subsystem is somewhat similar to the notion of localizable entanglement which was also defined based on local measurements on fixed subsystems of ρ (Verstraete and Popp, 2004; Popp et al., 2005).

This MIN measure is shown to have the basic properties: (i) $N_G(\rho)\geqslant 0$, and the inequality holds for any product state. (ii) it is locally unitary invariant, namely, $N_G(U_{AB}\rho U_{AB}^\dagger) = N_G(\rho), \forall\, U_{AB} = U_A\otimes U_B$. (iii) For the case of nondegenerate reduced state $\rho_A = \sum_k\lambda_k|k\rangle\langle k|$, the optimal $\tilde{\Pi}^A$ is given by $\tilde{\Pi}^A(\rho) = \sum_k|k\rangle\langle k|\rho|k\rangle\langle k|$.

For $(m\times n)$-dimensional states of Eq. (27), $N_G(\rho)$ is upper bounded by $\sum_{i=1}^{m^2-m}\lambda_i$, where λ_i $(i = 1, 2, \ldots, m^2-1)$ denote the eigenvalues of RR^T in nonincreasing order, $R = (r_{ij})$ with $i, j \geq 1$ is a real matrix.

This MIN measure can be derived analytically for a wide range of quantum states, which include the pure states, the bipartite states ρ_{AB} with A being a qubit, certain higher dimensional states with symmetry, as well as certain bound entangled

Please cite this article in press as: Hu M.-L., et al., Quantum coherence and geometric quantum discord. Physics Reports (2018), https://doi.org/10.1016/j.physrep.2018.07.004.

states (Rana and Parashar, 2013) and other special states with degenerate ρ_A (Mirafzali et al., 2013). Some of the results are summarized as follows:

(1) For pure state $|\psi\rangle$ with the Schmidt decomposition of Eq. (11), one has

$$N_G(|\psi\rangle\langle\psi|) = 1 - \sum_k \lambda_k^2. \tag{100}$$

(2) For bipartite state ρ of Eq. (27) with $\dim\mathcal{H}_A = 2$, one has

$$N_G(\rho) = \begin{cases} \|R\|_2^2 - \dfrac{1}{\|\vec{x}\|_2^2}\vec{x}^T RR^T \vec{x} & \text{if } \vec{x} \neq 0, \\ \|R\|_2^2 - \lambda_{\min}(RR^T) & \text{if } \vec{x} = 0. \end{cases} \tag{101}$$

where $\lambda_{\min}(RR^T)$ denotes the smallest eigenvalue of RR^T, and $\vec{x} = (r_{10}, r_{20}, r_{30})^T$.

(3) For ρ_W and ρ_I of Eq. (25), one has (Luo and Fu, 2010b)

$$\begin{aligned} N_G(\rho_W) &= \frac{(dx-1)^2}{d(d+1)(d^2-1)}, \\ N_G(\rho_I) &= \frac{(d^2x-1)^2}{d(d+1)(d^2-1)}. \end{aligned} \tag{102}$$

Guo and Hou (2013b) given a necessary and sufficient condition for nullity of the HS norm of MIN. Let ρ be a bipartite state acting on $\mathcal{H}_A \otimes \mathcal{H}_B$, and write $\rho = \sum_{i,j} A_{ij} \otimes |i^B\rangle\langle j^B|$ [similar to Eq. (46)], they showed that $N_G(\rho) = 0$ if and only if the A_{ij}s are mutually commuting normal operators, and each eigenspace of $\rho_A = \sum_i A_{ii}$ is contained in some eigenspace of A_{ij}, $\forall\, i, j$.

Furthermore, it is shown that for a zero-MIN state ρ with $\dim\mathcal{H}_A \geqslant 3$, any local channel acting on party A cannot create MIN if and only if either it is a completely contractive channel or it is a nontrivial isotropic channel (Guo and Hou, 2013b). For the qubit case this property is an additional characteristic of the completely contractive channel or the commutativity-preserving unital channel. That is, MIN can also be created under local operations and classical communication (LOCC).

2.2.2. *Trace norm of MIN*

Similar to the GQD measured with the HS norm, the flaw of $N_G(\rho)$ is that it is also noncontractive under CPTP maps. Explicitly, it can be increased or decreased by trivial local reversible operations on the unmeasured party B. For example, a map $\mathcal{E}_B(\rho) = \rho \otimes \rho_C$ leads to $N_G(\rho_{A:BC}) = N_G(\rho)\mathrm{tr}(\rho_C)^2$. As the purity $\mathrm{tr}\rho_C^2 \leq 1$, this equality means that the MIN is decreased by simply introducing an uncorrelated local ancillary system. As a matter of fact, the flaw of the HS norm of MIN, being noncontractive under CPTP maps, is the same flaw for every HS norm measure of correlation. Despite this flaw, the MIN defined originally using the HS norm, in particular its motivation, inspires one to introduce most of the subsequent MIN measures.

Motivated by using the trace norm to measure GQD (Paula et al., 2013a), Hu and Fan (2015a) proposed that this norm can also be used to measure MIN, with the explicit expression

$$N_T(\rho) = \max_{\Pi^A} \|\rho - \Pi^A(\rho)\|_1. \tag{103}$$

This definition, although amends slightly the definition of Eq. (98), avoids successfully its non-contractivity problem. One can show that $N_T(\rho)$ is nonincreasing under any CPTP map $\mathcal{E}_B$ (Hu and Fan, 2015a), i.e., $N_T(\rho) \geq N_T(\mathcal{E}_B[\rho])$. The proof is as follows: Let $\bar{\Pi}^A$ be the optimal measurement for obtaining $N_T(\rho)$, and $\tilde{\Pi}^A$ be the optimal measurement for obtaining $N_T(\mathcal{E}_B[\rho])$, then as $\mathcal{E}_B$ and $\tilde{\Pi}^A$ commute, we obtain $\tilde{\Pi}^A(\mathcal{E}_B[\rho]) = \mathcal{E}_B(\tilde{\Pi}^A[\rho])$, and therefore

$$\begin{aligned} N_T(\rho) &= \|\rho - \bar{\Pi}^A(\rho)\|_1 \\ &\geq \|\rho - \tilde{\Pi}^A(\rho)\|_1 \\ &\geq \|\mathcal{E}_B(\rho) - \mathcal{E}_B(\tilde{\Pi}^A[\rho])\|_1 \\ &= N_T(\mathcal{E}_B[\rho]), \end{aligned} \tag{104}$$

where the first inequality comes from the fact that $\tilde{\Pi}^A \neq \bar{\Pi}^A$ in general, and the second inequality is due to the contractivity of the trace norm under CPTP map. Therefore, $N_T(\rho)$ circumvents successfully the problem incurred for $N_G(\rho)$.

In purse of the analytical solutions of $N_T(\rho)$, some main results are as follows:

(1) For $(2\times n)$-dimensional state $|\psi\rangle$ with the Schmidt decomposition of Eq. (11), one has $N_T(|\psi\rangle\langle\psi|) = 2\sqrt{\lambda_1\lambda_2}$.

(2) For two-qubit state ρ decomposed as Eq. (27), with the addition $r_{ij} = 0$ for $i \neq j$, we have

$$N_T(\rho) = \begin{cases} \dfrac{\sqrt{\chi_+} + \sqrt{\chi_-}}{\|\vec{x}\|_1} & \text{if } \vec{x} \neq 0, \\ 2\max\{|r_{11}|, |r_{22}|, |r_{33}|\} & \text{if } \vec{x} = 0, \end{cases} \tag{105}$$

where the corresponding parameters are

$$\begin{aligned} &\chi_{\pm} = \alpha \pm 4\sqrt{\beta}|\vec{x}|,\ \alpha = |\vec{r}|^2|\vec{x}|^2 - |\vec{r}\cdot\vec{x}|^2, \\ &\vec{r} = (r_{11}, r_{22}, r_{33}),\ \beta = \sum_{\langle ijk\rangle} x_i^2 r_{jj}^2 r_{kk}^2, \end{aligned} \tag{106}$$

and the summation in the second line of the above equation runs over all the cyclic permutations of $\{1, 2, 3\}$.

(3) For ρ_W and ρ_I of Eq. (25), solutions of the trace norm MIN are given, respectively, by

$$N_T(\rho_W) = \frac{|dx-1|}{d+1},\ \ N_T(\rho_I) = \frac{2|d^2x-1|}{d(d+1)}, \tag{107}$$

and by comparing them with Eq. (102), one can see that for the present cases, the two MIN measures N_G and N_T give qualitatively the same descriptions of nonlocality.

2.2.3. Bures distance of MIN

By changing the maximization of Eq. (54), one can define the Bures distance of MIN as follows (Hu and Fan, 2015a)

$$N_B(\rho) = \max_{\Pi^A}\{1 - \sqrt{F(\rho, \Pi^A(\rho))}\}, \tag{108}$$

where Π^A is still the locally invariant measurements on party A, and $F(\rho,\sigma)$ is the Uhlmann fidelity defined in Eq. (52).

Compared with the former two measures of MIN, the calculation of the present MIN is more complicated. But when A is a qubit, the minimum Uhlmann fidelity $F_{\min}(\rho, \Pi^A[\rho]) = \min_{\Pi^A} F(\rho, \Pi^A[\rho])$ can be calculated via Eq. (56), with however, the maximization being replaced by the minimization.

For Bell-diagonal state ρ^{Bell} of Eq. (34), its square root can be derived explicitly, from which $F_{\min}(\rho^{\mathrm{Bell}}, \Pi^A[\rho^{\mathrm{Bell}}])$ can be calculated as

$$F_{\min} = \frac{1}{2}\left(1 + \min_{\{\theta,\phi\}} \sqrt{b_3^2 + (b_{13}^2 + b_{21}^2 \sin^2\phi)\sin^2\theta}\right), \tag{109}$$

where $b_{ij}^2 = b_i^2 - b_j^2$, $b_i = 8(t_0^2 + t_i^2) - 1$ $(i = 1, 2, 3)$, and by writing $c_{\mathrm{sum}} = c_1 + c_2 + c_3$, we have

$$\begin{aligned} t_0 =& \frac{1}{8}\sqrt{1 - c_{\mathrm{sum}}} + \frac{1}{8}\sum_{k=1}^{3}\sqrt{1 + c_{\mathrm{sum}} - 2c_k}, \\ t_i =& -\frac{1}{8}\sqrt{1 - c_{\mathrm{sum}}} + \frac{1}{8}\sum_{k=1}^{3}\sqrt{1 + c_{\mathrm{sum}} - 2c_k} \\ & - \frac{1}{4}\sqrt{1 + c_{\mathrm{sum}} - 2c_i}. \end{aligned} \tag{110}$$

From Eq. (110) one can see that $F_{\min}$ equals $(1 + |b_1|)/2$ if $|b_1| \leqslant \min\{|b_2|, |b_3|\}$, $(1 + |b_2|)/2$ if $|b_2| \leqslant \min\{|b_1|, |b_3|\}$, and $(1 + |b_3|)/2$ otherwise.

2.2.4. Relative entropy of MIN

The relative entropy can also be recognized as a (pseudo) distance measure of quantum states, though technically it does not has a geometric interpretation as it is not symmetric, i.e., $S(\rho \parallel \sigma) \neq S(\sigma \parallel \rho)$ in general. It has been used to define the relative entropy of discord and quantum dissonance (Modi et al., 2010).

Xi and Wang (2012) introduced the relative entropy of MIN as

$$N_R(\rho) = \max_{\Pi^A} S(\rho \parallel \Pi^A[\rho]), \tag{111}$$

where $\Pi^A(\rho) = \sum_i \Pi_i^A \rho \Pi_i^A$, and $\{\Pi_i^A\}$ is the set of locally invariant projective measurements.

This MIN measure has been showed to be well defined. It possesses the same basic properties (i), (ii), and (iii) as that of the HS norm of MIN. Furthermore, $N_R(\mathcal{E}_B[\rho]) \leq N_R(\rho)$ for any CPTP map $\mathcal{E}_B$ on the unmeasured party B (Hu and Fan, 2012a). It is also intimately related to the HS norm of MIN, $N_E(\rho) \geq N_G^2(\rho)/(2\ln 2)$ (Xi and Wang, 2012).

It vanishes for the classical–quantum state χ with nondegenerate reduced density operator $\chi_A = \mathrm{tr}_B\chi$, or for χ with degenerate χ_A and $\rho_k^B = \rho_l^B$ $(\forall\ k, l)$, see Eq. (15). Moreover, it is lower bounded by $-S(A|B)$ and upper bounded by $\min\{I(\rho), S(\rho_A)\}$, with $S(A|B) = S(\rho) - S(\rho_B)$ the conditional entropy (Hu and Fan, 2012a). For ρ^{Bell} of Eq. (34), analytical solution of it is given by

$$\begin{aligned} N_R(\rho^{\text{Bell}}) =& 1+H\left(\frac{1+c_-}{2}\right)+\frac{1-c_{\text{sum}}}{4}\log_2\frac{1-c_{\text{sum}}}{4} \\ &+\sum_{k=1}^{3}\frac{1+c_{\text{sum}}-2c_k}{4}\log_2\frac{1+c_{\text{sum}}-2c_k}{4}, \end{aligned} \tag{112}$$

with $H(\cdot)$ being the binary Shannon entropy function.

The measure $N_R(\rho)$ is equivalent to that of the entropic MIN defined as the maximal discrepancy between QMI of the pre- and post-measurement states as (Hu and Fan, 2012a)

$$N_E(\rho) = I(\rho) - \min_{\Pi^A} I[\Pi^A(\rho)], \tag{113}$$

where $I(\rho)$ is the QMI given by Eq. (1).

This MIN quantifies in fact, the maximal loss of total correlations under locally non-disturbing measurements Π^A. Moreover, as ρ and $\Pi^A(\rho)$ have the same reduced states, $N_E(\rho)$ defined above is equivalent to

$$N_E(\rho) = \max_{\Pi^A} S(\Pi^A[\rho]) - S(\rho). \tag{114}$$

Thus, this measure of MIN quantifies also the maximal increment of von Neumann entropy induced by Π^A. Moreover, as the entropy of a state measures how much uncertainty there is in it, $N_E(\rho)$ can also be interpreted as the maximal increment of our uncertainty about the considered system induced by the locally invariant measurements.

2.2.5. *Skew information measure of MIN*

Apart from measuring uncertainty in a state, the WY skew information has also been proposed to measure MIN. Its definition is as follows (Li et al., 2016)

$$N_{SI}(\rho) = \max_{\tilde{K}^A} \sum_{i=1}^{m} \mathrm{I}(\rho, \tilde{K}_i^A \otimes \mathbb{1}_B), \tag{115}$$

which is in some sense dual to the correlation measure given in Eq. (86), with however the rank-one projectors $\tilde{K}^A = \{\tilde{K}_i^A\}$ are restricted to those which do not disturb $\rho_A = \mathrm{tr}_B\rho$.

This MIN measure is invariant under locally unitary operations, contractive under CPTP map $\mathcal{E}_B$ on party B, and vanishes for all the product states and the classical–quantum states with nondegenerate reduced state ρ_A. For general state the calculation of $N_{SI}(\rho)$ is difficult. But if we decompose $\sqrt{\rho}$ as Eq. (73), an upper bound can be obtained as follows (Li et al., 2016)

$$N_{SI}(\rho) \le 1 - \sum_{i=1}^{m-1} \mu_i, \tag{116}$$

with μ_i $(i = 1, 2, \ldots, m^2)$ being the eigenvalues of $\Gamma\Gamma^T$ listed in decreasing order (counting multiplicity), and $\Gamma = (\gamma_{ij})$ is the $(m^2 \times n^2)$-dimensional correlation matrix.

For the pure states $\psi = |\psi\rangle\langle\psi|$, $N_{SI}(\psi) = N_G(\psi)$, while for the bipartite states ρ with A being a qubit, one has $N_{SI}(\rho) = 1 - \mu_1$ if $\vec{u} = 0$, and

$$N_{SI}(\rho) = 1 - \frac{1}{2}\mathrm{tr}\left(\begin{pmatrix} 1 & \vec{u}_0 \\ 1 & -\vec{u}_0 \end{pmatrix} \Gamma\Gamma^T \begin{pmatrix} 1 & \vec{u}_0 \\ 1 & -\vec{u}_0 \end{pmatrix}^T\right), \tag{117}$$

if $\vec{u} \neq 0$. Here, $\vec{u} = (u_1, u_2, u_3)$ with $u_i = \mathrm{tr}(\rho_A\sigma_i)/\sqrt{2}$, and $\vec{u}_0 = \vec{u}/|\vec{u}|$. Moreover, for ρ_W and ρ_I of Eq. (34), one has

$$\begin{aligned} N_{SI}(\rho_W) &= \frac{1}{2}\left(\frac{d-x}{d+1} - \sqrt{\frac{d-1}{d+1}(1-x^2)}\right), \\ N_{SI}(\rho_I) &= \frac{1}{d}\left(\sqrt{(d-1)x} - \sqrt{\frac{1-x}{d+1}}\right)^2. \end{aligned} \tag{118}$$

Similar to the above measure, Wu et al. (2014) introduced another MIN-like nonlocality measure which was termed as uncertainty-induced nonlocality. It takes the form

$$\mathcal{U}_{SI}(\rho) = \max_{K^A} \mathrm{I}(\rho, K^A \otimes \mathbb{1}_B), \tag{119}$$

where K^A is a Hermitian observable with nondegenerate spectrum, and $[K^A, \rho_A] = 0$. This measure is locally unitary invariant, nonincreasing under any CPTP map on the unmeasured party B. Moreover, it can also be interpreted by the Hellinger distance via the equality

$$\mathcal{U}_{SI}(\rho) = \max_{K^A} d_H^2(\rho, K^A\rho K^A). \tag{120}$$

For $(2 \times n)$-dimensional state of Eq. (27), the uncertainty-induced nonlocality can be obtained explicitly as

$$\mathcal{U}_{SI}(\rho) = \begin{cases} 1 - \lambda_{\min}(W_{AB}) & \text{if } \vec{x} = 0, \\ 1 - \dfrac{1}{|\vec{x}|^2}\vec{x}^T W_{AB}\vec{x} & \text{if } \vec{x} \neq 0, \end{cases} \tag{121}$$

where $\vec{x} = (r_{10}, r_{20}, r_{30})^T$, and $\lambda_{\min}(W)$ is the smallest eigenvalue of the 3×3 matrix W_{AB}, the elements of which is given by Eq. (81).

2.2.6. Generalization of the MIN measures

The MIN measures we reviewed in the above sections reveal in fact only partial information about nonlocal features of a state, as they are defined based on the one-sided locally invariant measurements, thus those measures are all asymmetric. But a local state with respect to one party may be nonlocal with respect to another party. From this respect of view, it is significant to extend their definitions to more general case of two-sided locally invariant measurements. This gives the symmetric measure of MIN which can be written as

$$\tilde{N}(\rho) = \max_{\tilde{\delta} \in \mathcal{L}} D(\rho, \tilde{\delta}), \tag{122}$$

with $\tilde{\delta}$ being the two-sided locally invariant states in the sense that $\Pi^{AB}\tilde{\delta}\Pi^{AB} = \tilde{\delta}$ (with $\Pi^{AB} = \Pi^A \otimes \Pi^B$) should be satisfied, and $\sum_k \Pi_k^A \rho_A \Pi_k^A = \rho_A$ and $\sum_k \Pi_k^B \rho_B \Pi_k^B = \rho_B$ for any bipartite state ρ.

As an explicit example, we list the symmetric MIN measure defined based on the HS norm, i.e., $\tilde{N}_G(\rho) = \max_{\Pi^{AB}} \|\rho - \Pi^{AB}(\rho)\|_2^2$. This measure is locally unitary invariant, and vanishes for the product states. For the pure state $\psi = |\psi\rangle\langle\psi|$, we have $\tilde{N}_G(\psi) = N_G(\psi)$ (Guo, 2013). In fact, $\tilde{N}_G(\rho)$ can also be extended to N-partite quantum states. The definition can be written in the same form of Eq. (122), with however the locally invariant measurements $\Pi^{A_1} \otimes \Pi^{A_2} \otimes \ldots \otimes \Pi^{A_N}$, with $\sum_k \Pi_k^{A_i} \rho_{A_i} \Pi_k^{A_i} = \rho_{A_i}$ for $i = \{1, 2, \ldots, N\}$, and ρ_{A_i} the reduced state of the subsystem A_i. But now the evaluation of their analytical expression becomes a hard work.

2.3. Applications of geometric quantum discord

Up to now, we have presented an overview of the formal definitions and related formulae of the discordlike correlations defined via different distances. In general, these measures are conceptually different, and it is natural to wonder in what context one is more or less useful than the other. As a matter of fact, these measures capture different characteristic features of a state, and may have different physical implications and potential applications, e.g., the LQU (equivalent to the Hellinger distance of discord for any two-qubit state) guarantees a minimum precision of phase estimation (Girolami et al., 2013), the GQD defined via the relative entropy enables a direct comparison of it with the relative entropy of entanglement, and the negativity of quantumness can be connected to the negativity of entanglement. The above correlations defined with different distances may play role in different quantum information protocols, e.g., the HS norm of discord bounds from above fidelity of quantum teleportation and remote state preparation. Moreover, these discordlike correlations may reveal different aspects of the physical properties of a many-body system, and this will be discussed in Section 7 of this review.

2.3.1. Quantum teleportation

To teleport a state from one party to another spatially separated party, the sender Alice and the receiver Bob should share a quantum channel ρ, and one can achieve a perfect teleportation if ρ is maximally entangled (Bennett et al., 1993). However, entanglement of ρ is the prerequisite but not the only key elements for accomplishing the teleportation protocol. This is because for the non-maximally entangled channel, the fidelity of teleportation is not proportional to the amount of entanglement in ρ, e.g., it has been showed that the purity of ρ is also a crucial element in determining the quality of the teleportation protocol (Hu, 2011).

When the channel is composed of a general two-qubit state ρ as given by Eq. (22), the average teleportation fidelity, based on the assumption that Bob can perform all kinds of recovery operations to his qubit, can be derived as $\bar{F} = 1/2 + \mathrm{tr}\sqrt{R^\dagger R}/6$ (Horodecki et al., 1996). By considering a normalized version of the HS norm of discord

$$\tilde{D}_G(\rho) = \frac{d}{d-1} D_G(\rho), \tag{123}$$

Satyabrata and Subhashish (2012) identified a connection between an upper bound of $\tilde{D}_G(\rho)$ and $\bar{F}(\rho)$. The bound of $\tilde{D}_G(\rho)$ was derived by using the Weyl's theorem, and is given by

$$\tilde{D}_G^{\max}(\rho) = \frac{1}{3}\left[\|R^2\| - k_{\max}(RR^T)\right], \tag{124}$$

where $k_{\max}(RR^T)$ represents the largest eigenvalue of RR^T. As $\mathrm{tr}\sqrt{R^\dagger R} \geqslant \|R^2\|$, one can show that

$$\bar{F}(\rho) \geqslant \frac{1+\tilde{D}_G^{\max}(\rho)}{2}. \tag{125}$$

On the other hand, by using the relations $3\bar{F}(\rho) - 2 \leq \mathcal{N}(\rho)$ (Verstraete and Verschelde, 2002) and $\mathcal{N}^2(\rho) \leq \tilde{D}_G(\rho)$ (Girolami and Adesso, 2011) for all two-qubit states, with $\mathcal{N}(\rho)$ being an entanglement measure called negativity (Vidal and Werner, 2002), one can obtain

$$\bar{F}(\rho) \leqslant \frac{2+\sqrt{\tilde{D}_G(\rho)}}{3}. \tag{126}$$

These two equations show that the GQD bounds the average teleportation fidelity. But a direct quantitative connection between the various discordlike quantum correlation measures and $\bar{F}$ does not exist.

2.3.2. Remote state preparation

Remote state preparation (RSP) is a quantum protocol for remotely preparing a quantum state by LOCC (Bennett et al., 2001). To accomplish this task, the two participants, Alice and Bob, also need to share a correlated channel. But different from the protocol of quantum teleportation (Bennett et al., 1993), Alice knows what state to be transmitted in advance, so the amount of required classical information can be reduced.

If the shared state is maximally entangled, one can accomplish a perfect state preparation. Otherwise, the fidelity of the protocol may be reduced. Dakić et al. (2012) considered such a problem. They considered the channel to be a general two-qubit state of the form of Eq. (22), and Alice wants to prepare a qubit state $\rho(\vec{s}) = (I + \vec{s}\cdot\sigma)/2$ with the Bloch vector $\vec{s}$ in the plane orthogonal to the direction $\vec{\beta}$. To this purpose, she performs the local measurements $\Pi_\alpha^A = [I + \alpha\vec{\alpha}\cdot\vec{\sigma}]/2$ along the direction $\vec{\alpha}$ and informs Bob of her outcome $\alpha = \pm 1$. The Bloch vector of Bob's state can then be obtained as

$$\vec{y}_\alpha = \frac{\vec{y} + \alpha R^T\vec{\alpha}}{1 + \alpha\vec{x}\cdot\vec{\alpha}}. \tag{127}$$

If Alice's outcome is $\alpha = -1$, Bob applies a π rotation about $\vec{\beta}$ to his system, whereas no operation is required for $\alpha = 1$. After these conditional operations, the Bloch vector of Bob's resulting state becomes the following mixture

$$\vec{r} = p_+\vec{y}_+ + p_- R(\pi)\vec{y}_-. \tag{128}$$

where $p_\alpha = (1 + \alpha\vec{\alpha}\cdot\vec{x})/2$ is the probability for Alice's measurement outcome α.

To evaluate the efficiency of the RSP protocol, Dakić et al. (2012) defined the payoff-function $\mathcal{P} = (\vec{r}\cdot\vec{s})^2$ which is proportional to the fidelity $F = \mathrm{tr}[\rho(\vec{r})\rho(\vec{s})] = (1 + \vec{r}\cdot\vec{s})/2$. For the present case, $\mathcal{P}$ can be derived explicitly as

$$\mathcal{P} = (\vec{\alpha}^T R\vec{s})^2 = \sum_{j=1}^{3}[\alpha_j(r_{j1}s_1 + r_{j2}s_2 + r_{j3}s_3)]^2, \tag{129}$$

and by optimizing over Alice's choice of $\vec{\alpha}$, one can obtain

$$\mathcal{P}_{opt} = \sum_{j=1}^{3}(r_{j1}s_1 + r_{j2}s_2)^2. \tag{130}$$

Finally, the expected payoff is averaged over the distribution $\vec{s}$ and minimized over all possible choices of $\vec{\beta}$. The corresponding RSP-fidelity is given by

$$\mathcal{F} = \frac{1}{2}(E_2 + E_3), \tag{131}$$

where $E_1 \geqslant E_2 \geqslant E_3$ are the eigenvalues of $R^T R$ arranged in nonincreasing order. Clearly, $\mathcal{F}$ vanishes if and only if $E_2 = E_3 = 0$, which corresponds to a zero-discord state.

Moreover, if the local Bloch vector $\vec{x}$ of ρ_{AB} is parallel to the eigenvector corresponding to largest eigenvalue of $R^T R$, the HS norm of discord is given by $D_G = \mathcal{F}/2$ (Dakić et al., 2012), which endows the GQD an operational interpretation.

Note that the nonvanishing GQD in the channel state is a necessary but not sufficient condition for RSP, as it has been found that there are discordant states which yields zero RSP-fidelity, e.g., the family of two-qubit states described by the real density matrix with $\rho_{11} - \rho_{22} = \rho_{44} - \rho_{33}$, $\rho_{14} = \rho_{23} = 0$, and $\rho_{12} = \rho_{13} = \rho_{24} = \rho_{34}$ (Giorgi, 2013).

2.3.3. Phase estimation

Girolami et al. (2013) considered a phase estimation task in which a bipartite state ρ is utilized as a probe. In this task, a local unitary operation U_ϕ is performed on subsystem A of this system, therefore an unknown phase ϕ is encoded to it and ρ is transformed to $\rho_\phi = (U_\phi \otimes \mathbb{1})\rho(U_\phi^\dagger \otimes \mathbb{1})$. One's goal is to estimate as precisely as possible the parameter ϕ. For a given probe state ρ, one can optimize the measurements performed on ρ_ϕ to achieve the Cramér–Rao bound (Giovannetti et al., 2011)

$$V\,ar(\tilde{\phi}_{\text{best}}) = \frac{1}{N\mathcal{F}(\rho_\phi)}, \tag{132}$$

where $V\,ar(\tilde{\phi}_{\text{best}})$ is the variance of the best unbiased estimator $\tilde{\phi}_{\text{best}}$, N is the times of independent measurements, and

$$\mathcal{F}(\rho_\phi) = \mathrm{tr}(\rho_\phi L_\phi^2), \tag{133}$$

is the quantum fisher information, with L_ϕ being the symmetric logarithmic derivative determined by

$$\frac{\partial \rho_\phi}{\partial \phi} = \frac{1}{2}(L_\phi \rho_\phi + \rho_\phi L_\phi). \tag{134}$$

For the above phase estimation task, Girolami et al. (2013) proved that

$$V\,ar(\tilde{\phi}_{\text{best}}) \leq \frac{1}{4N\mathcal{U}_A^\Lambda(\rho)}, \tag{135}$$

hence the inverse of the LQU limits the achievable precision of the estimated phase parameter ϕ. This gives an operational interpretation of the LQU.

3. Quantum coherence measures

Different from quantum correlations which are defined in the framework of bipartite and multipartite scenarios, quantum coherence is related to the characteristics of the whole system. In general, the starting point for the resource theoretic characterization of a quantum character, e.g., quantum entanglement (Horodecki et al., 2009) and QD (Modi et al., 2012), is the identification of free states which can be created at no cost and free operations which transform any free state into free state.

In a manner similar to the resource framework of entanglement where the free states are identified as those of the separable one and the free operations are specified by the LOCC, the set $\mathcal{I}$ of free states for quantum coherence encompasses those of the incoherent states which are diagonal in the prefixed reference basis $\{|i\rangle\}_{i=1}^d$, and take the form (Baumgratz et al., 2014)

$$\delta = \sum_{i=1}^{d} \delta_i |i\rangle\langle i|, \tag{136}$$

for a d-dimensional Hilbert space.

Within the framework of Baumgratz et al. (2014), the set of free operations are those of the incoherent operations (IO) which can be specified by the Kraus operators $\{K_i\}$ satisfying $\sum_i K_i^\dagger K_i = \mathbb{1}$. Based on the measurements with and without subselection, Baumgratz et al. (2014) further identified two different classes of IO:

(A) The incoherent completely positive and trace preserving (ICPTP) operations which act as $\Lambda(\rho) = \sum_i K_i \rho K_i^\dagger$. Here, all K_i are of the same dimension, and should obey the property $K_i \delta K_i^\dagger / p_i \in \mathcal{I}$ for arbitrary $\delta \in \mathcal{I}$, with $p_i = \mathrm{tr}(K_i \rho K_i^\dagger)$ being the probability for obtaining the result i.

(B) The incoherent operations with subselection for which the output measurement results are retained. They also require $K_i \delta K_i^\dagger / p_i \in \mathcal{I}$ to be satisfied for any $\delta \in \mathcal{I}$. But the dimension of K_i may be different, that is, different K_i may correspond to different output spaces.

In general, a Kraus operator for an IO can be represented as $K_i = \sum_i c_i |f(i)\rangle\langle i|$, with the coefficient $c_i \in [0, 1]$ and $f(i)$ a function on the index set (Winter and Yang, 2016).

As showed through explicit examples by Shao et al. (2015) and proved strictly by Yao et al. (2015), the Kraus operators related to incoherent operations Λ are very limited. There is at most one nonzero entry in every column of K_i, and the number of possible structure of K_i (a legal structure stands for a possible arrangement of nonzero entries in K_i) is m^n for K_i being the $m \times n$ matrices. Streltsov et al. (2017b) further discussed the problem relevant to the number of Kraus operators in a general quantum operation. For a system of dimension d, it has been found that any IO admits a decomposition with at most $d^4 + 1$ Kraus operators. For $d = 2$ and 3, this number can be improved to 5 and 39, respectively.

Equipped with the sets of incoherent states and IO, Baumgratz et al. (2014) presented the defining conditions for a faithful coherence measure $C(\rho)$ which is a function that maps state ρ to a nonnegative real value:

(C1) Nonnegativity, i.e., $C(\rho) \geq 0$, and $C(\delta) = 0$ iff $\delta \in \mathcal{I}$.

(C2a) Monotonicity under ICPTP map, $C(\rho) \geq C(\Lambda[\rho])$.

(C2b) Monotonicity under selective IO on average, that is, $C(\rho) \geq \sum_i p_i C(\rho_i)$.

(C3) Convexity under mixing of states, i.e., $\sum_i p_i C(\rho_i) \geq C(\sum_i p_i \rho_i)$, with $\{p_i\}$ being the probability distribution.

Note that condition (C2b) is stronger than (C2a), as its combination with (C3) automatically imply (C2a). In general, a real-valued function $C(\rho)$ is called a coherence measure if it satisfies the above four conditions. If only the conditions (C1), (C2a), and (C2b) are satisfied, $C(\rho)$ is usually called a coherence monotone.

A dual notion to incoherent states is the maximally coherent state, which can serve as a unit for defining coherence measure (Baumgratz et al., 2014). It takes the form

$$|\Psi_d\rangle = \frac{1}{\sqrt{d}}\sum_{i=1}^{d}|i\rangle, \tag{137}$$

for which any other ρ in the same Hilbert space can be generated with certainty by merely IO on it.

Du et al. (2015) considered problem of general pure states transformation under IO by using the majorization theory (Bhatia, 1997). For states $|\psi\rangle = \sum_{i=1}^{d}\psi_i|i\rangle$ and $|\phi\rangle = \sum_{i=1}^{d}\phi_i|i\rangle$, with the parameters $\{|\psi_i|\}$ and $\{|\phi_i|\}$ being arranged in nonincreasing order, they proved that $|\psi\rangle$ can be transformed to $|\phi\rangle$ via IO if and only if $(|\psi_1|^2, |\psi_2|^2, \ldots, |\psi_d|^2)^T$ is majorized by $(|\phi_1|^2, |\phi_2|^2, \ldots, |\phi_d|^2)^T$, i.e.,

$$(|\psi_1|^2, |\psi_2|^2, \ldots, |\psi_d|^2)^T \prec (|\phi_1|^2, |\phi_2|^2, \ldots, |\phi_d|^2)^T. \tag{138}$$

Moreover, by applying the general unitary incoherent operations

$$U_I = \sum_j e^{i\theta_j}|\alpha_j\rangle\langle j| \tag{139}$$

on $|\Psi_d\rangle$, with $\{\alpha_j\}$ being a relabeling of $\{j\}$, Peng et al. (2016) found that the complete set $\mathcal{M}$ of maximally coherent states is composed of $\rho^{\rm mcs} = |\Psi_d^{\rm mcs}\rangle\langle\Psi_d^{\rm mcs}|$, with

$$|\Psi_d^{\rm mcs}\rangle = \frac{1}{\sqrt{d}}\sum_j e^{i\theta_j}|j\rangle. \tag{140}$$

Building upon this, they proposed that U_I are the unique quantum operations that preserve the coherence of a state, and suggested an additional condition for a valid coherence measure, i.e., (C4) $C(\rho)$ should assign a maximal value only to $\rho^{\rm mcs}$.

Yu et al. (2016c) proposed an alternative framework for defining coherence, in which their first two conditions are the same as (C1) and (C2a), while (C2b) and (C3) are replaced by one condition, that is, the additivity requirement of coherence for subspace-independent states. To be precise, for ρ_1 and ρ_2 in two different subspaces, the amount of coherence contained in $\rho = p_1\rho_1 \oplus p_2\rho_2$ (with p_1 and p_2 being probabilities) should be neither more nor less than the average coherence of ρ_1 and ρ_2 due to the block-diagonal structure of ρ. Hence, a reasonable measure of coherence should satisfy the condition

$$C(p_1\rho_1 \oplus p_2\rho_2) = p_1C(\rho_1) + p_2C(\rho_2), \tag{141}$$

the above condition, together with (C1) and (C2a), have been shown to be equivalent to the four conditions introduced by Baumgratz et al. (2014).

While the set of free or incoherent states is widely accepted, there is no general consensus on the set of free operations in the resource theory of coherence. Apart from the above mentioned IO, there are other forms of free operations being introduced based on different physical or mathematical motivations up to date. Three typical ones are as follows:

(1) Maximally incoherent operations (MIO). It refers to the set of physically realizable quantum operations Φ which maps incoherent states into incoherent states, i.e., $\Phi(\mathcal{I}) \in \mathcal{I}$ (Aberg, 2006). Obviously, this is the most general class of operations which do not create coherence from incoherent states.

(2) Dephasing-covariant incoherent operations (DIO). The relevant set of it is a subset of MIO with the additional property $[\Delta, \Phi] = 0$ (Chitambar and Gour, 2016a,b; Marvian and Spekkens, 2016). That is, it admits $\Lambda[\Delta(\rho)] = \Delta[\Lambda(\rho)]$.

(3) Strictly incoherent operations (SIO). This type of operations also admit an incoherent Kraus decomposition $\{K_i\}$ for which not only K_i but also $K_i^\dagger$ ($\forall\, i$) is incoherent (Winter and Yang, 2016). That is, $\Delta(K_i\rho K_i^\dagger) = K_i\Delta(\rho)K_i^\dagger$, where

$$\Delta(\rho) = \sum_i \langle i|\rho|i\rangle|i\rangle\langle i|, \tag{142}$$

denotes full dephasing of ρ in the basis $\{|i\rangle\}_{i=1}^d$. It is the most general class of operations which do not use coherence, and admits a decomposition with at most $\min\{d^4+1, \sum_{k=1}^{d} d!/(k-1)!\}$ Kraus operators (Streltsov et al., 2017b).

The inclusion relation of the above free operations are given by SIO $\subset$ IO $\subset$ MIO and SIO $\subset$ DIO $\subset$ MIO.

The definitions of incoherent state δ and maximally coherent state $|\Psi_d\rangle$ imply that the related coherence measure will be a basis dependent quantity. This is because any density operator can be diagonalized in the reference basis spanned by its eigenvectors, hence casting a doubt on the rationality of this framework. But the recent progresses, particularly those studied from an operational perspective, still yields physically meaningful results. Moreover, in practice the reference basis is usually

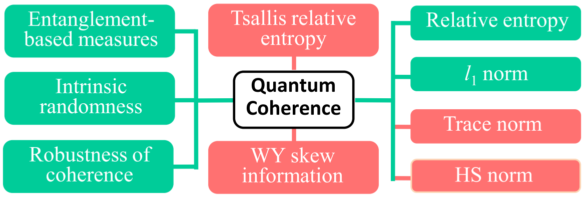

Fig. 3. Different quantum coherence measures. Here, the measures based on relative entropy, l_1 norm, entanglement, intrinsic randomness, and the robustness of coherence satisfy all the required conditions, while the others satisfy only partial of them.

chosen according to the physical problem under consideration. All these indicate that the study of coherence measure has its own irreplaceable role.

Up to now, there are a number of quantum coherence measures being proposed in the literature (see Fig. 3), where some of them satisfy the defining conditions, while the others satisfy only partial of these conditions. We review them in detail in the following.

3.1. Distance-based measures of coherence

With the advent of quantum information science, geometric approaches are used to treat a huge class of problems such as the characterization and quantification of various quantum features. Analogously to the resource theory of entanglement for which the free operations are described by LOCC, the free states correspond to the separable states, and the entanglement can be defined by a distance between the considered state and the set of separable states, it is natural to quantify coherence of a state by utilizing a distance measure because coherence is also placed in a resource theoretic framework. To be explicit, one can quantify the amount of coherence contained in a state ρ by using the minimal distance between ρ and the set $\mathcal{I}$ of incoherent states, i.e.,

$$C_{\mathcal{D}}(\rho) = \min_{\delta\in\mathcal{I}} \mathcal{D}(\rho, \delta), \tag{143}$$

where $\mathcal{D}(\rho, \delta)$ denotes certain distance measures of quantum states.

By its definition of Eq. (143), the condition (C1) is fulfilled for the distance measure which gives $\mathcal{D}(\rho, \delta) = 0$ if and only if $\rho = \delta$, while (C2a) can be fulfilled when $\mathcal{D}$ is monotonous under the action of CPTP maps, i.e., $\mathcal{D}(\rho, \delta) \geq \mathcal{D}(\Lambda[\rho], \Lambda[\delta])$. Moreover, (C3) is also fulfilled if $\mathcal{D}$ is jointly convex, i.e., $\mathcal{D}(\sum_i p_i\rho_i, \sum_i p_i\sigma_i) \leq \sum_i p_i\mathcal{D}(\rho_i, \sigma_i)$.

3.1.1. Relative entropy of coherence

The relative entropy has been adopted to quantify entanglement, QD, and MIN. Baumgratz et al. (2014) showed that it can also serve as a valid tool for quantifying coherence. To be explicit, they defined

$$C_r(\rho) = \min_{\delta\in\mathcal{I}} S(\rho \parallel \delta) = S(\rho_{\rm diag}) - S(\rho), \tag{144}$$

where $\rho_{\rm diag}$ denotes the diagonal part of ρ.

This is an entropic measure of coherence which has a clear physical interpretation, as $C_r(\rho)$ equals the optimal rate of the distilled maximally coherent states by IO in the asymptotic limit of many copies of ρ (Winter and Yang, 2016).

For the one-qubit state $\rho = (\mathbb{1}_2 + \vec{r}\cdot\vec{\sigma})/2$, with $\vec{r}\in\mathbb{R}^3$ and $|\vec{r}| \leq 1$, and $\vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ is the vector of Pauli matrices, if one chooses the reference basis as the eigenbasis of $\hat{n}\cdot\vec{\sigma}$ ($\hat{n}$ is a unit vector in $\mathbb{R}^3$), then

$$C_r(\rho) = H\left(\frac{1+\vec{r}\cdot\hat{n}}{2}\right) - H\left(\frac{1+\vec{r}}{2}\right), \tag{145}$$

and $H(\cdot)$ is the binary Shannon entropy function.

For the two-qubit Bell-diagonal states of Eq. (34), Bromley et al. (2015) found that the closest incoherent state δ with respect to the bona fide distance measure of coherence (e.g., the relative entropy of coherence) is still a Bell-diagonal state with vanishing local Bloch vectors along x and y directions, and for the particular case of $c_2 = -c_1c_3$, δ reduces to the diagonal part of $\rho^{\rm Bell}$.

Hu and Fan (2017) and Yao et al. (2016b) studied maximal coherence of a state ρ under generic reference basis. When the dimension of ρ is d, they showed that the maximal coherence is given by

$$C_r^{\max}(\rho) = \log_2 d - S(\rho). \tag{146}$$

Yao et al. (2016b) also derived the corresponding unitary operator which transforms the computational basis to the optimal basis such that the maximal $C_r^{\max}(\rho)$ is obtained. It is given by $U = VH^\dagger$, with the column vectors of V being the eigenvectors of ρ, and H is the rescaled complex Hadamard matrices. In fact, the Fourier matrix (a subset of the complex Hadamard matrices) $\mathbf{F}_d$ with elements $[\mathbf{F}_d]_{\mu\nu} = e^{i2\pi\mu\nu/d}/\sqrt{d}$ is suffice for this purpose.

3.1.2. l_1 norm of coherence

Intuitively, the superposition corresponds to the nonvanishing off-diagonal elements of the density operator description of a quantum state with respect to the selected reference basis. Starting from this consideration, Baumgratz et al. (2014) showed that the l_1 norm can also serve as a bona fide measure of coherence. To be explicit, they defined it as

$$C_{l_1}(\rho) = \min_{\delta\in\mathcal{I}} \|\rho - \delta\|_{l_1} = \sum_{i\neq j} |\langle i|\rho|j\rangle|, \tag{147}$$

which equals sum of the absolute values of the off-diagonal elements of ρ, and is favored for its ease of evaluation. Apart from convexity, it also satisfy the inequality $C_{l_1}(p_1\rho_1+p_2\rho_2) \geq |p_1C_{l_1}(\rho_1)-p_2C_{l_1}(\rho_2)|$ (Dai et al., 2017). Moreover, for any bipartite pure state $|\psi\rangle_{AB}$, the l_1 norm of coherence equals twice of its negativity which is a measure of quantum entanglement (Vidal and Werner, 2002).

For pure state $\psi = |\psi\rangle\langle\psi|$, a relation between $C_{l_1}(\psi)$ and $C_r(\psi)$ has also been established (Rana et al., 2016), which is given by

$$C_{l_1}(\psi) \geq \max\{C_r(\psi), 2^{C_r(\psi)} - 1\}, \tag{148}$$

where $C_{l_1}(\psi)$ equals $C_r(\psi)$ if and only if the diagonal elements of ψ are (up to permutation) either $\{1, 0, \dots, 0\}$ or $\{1/2, 1/2, 0, \dots, 0\}$. It has also been proven that (Rana et al., 2017)

$$C_{l_1}^2(\psi) \leq \frac{d(d-1)C_r(\psi)}{\sqrt{2}}, \tag{149}$$

where $d = \mathrm{rank}(\psi)$ is the rank of ψ. If $d > 2$, one can further obtain $C_{l_1}(\psi) - C_r(\psi) \leq d - 1 - \log_2 d$. Rana et al. (2017) also proved a sharpest bound of $C_r(\psi)$ in terms of $C_{l_1}(\psi)$ as follows:

$$\begin{aligned} H(\alpha) + (1-\alpha)\log_2(d-1) &\leq C_r(\psi) \\ &\leq H(\beta) + (1-\beta)\log_2(n-1), \end{aligned} \tag{150}$$

where n equals $C_{l_1} + 1$ if C_{l_1} is an integer, and $[C_{l_1}] + 2$ ($[C_{l_1}]$ is the integer part of C_{l_1}) otherwise. By denoting $x = C_{l_1} + 1$, the other two parameters are given by

$$\begin{aligned} \alpha &= \frac{2 + (d-2)(d - C_{l_1}) + 2\sqrt{(d-1)(d-x)x}}{d^2}, \\ \beta &= \frac{2 + (n-2)(n - C_{l_1}) - 2\sqrt{(n-1)(n-x)x}}{n^2}. \end{aligned} \tag{151}$$

For general states ρ, one has

$$C_{l_1}(\rho) \geq C_r(\rho)/\log_2 d. \tag{152}$$

Rana et al. (2016) have also conjectured $C_{l_1}(\rho) \geq C_r(\rho)$, but it was proved only for pure states, single-qubit states, and pseudopure states $\rho = p\psi + (1-p)\delta$ ($\forall\delta \in \mathcal{I}$ and $p \in [0, 1]$), while for general case, one can only prove $C_{l_1}(\rho) \geq 2^{C_r(\rho)} - 1$ (Rana et al., 2017), which is somewhat similar to the case for pure states showed in Eq. (148).

Moreover, for any single-qubit state ρ, by using convexity of C_r and the inequality $2\min\{x, 1-x\} \leq H(x) \leq 2\sqrt{x(1-x)}$, $\forall x \in [0, 1]$, Rana et al. (2017) further proved the following relation

$$\begin{aligned} 1 - H\left(\frac{1 - C_{l_1}(\rho)}{2}\right) &\leq C_r(\rho) \leq H\left(\frac{1 - \sqrt{1 - C_{l_1}^2(\rho)}}{2}\right) \\ &\leq C_{l_1}(\rho), \end{aligned} \tag{153}$$

where the equality holds when ρ is either incoherent or maximally coherent.

In fact, any state ρ can be decomposed as

$$\rho = \frac{1}{d}\mathbb{1}_d + \frac{1}{2}\sum_{i=1}^{d^2-1} x_i X_i, \tag{154}$$

where $x_i = \mathrm{tr}(\rho X_i)$, and $\{X_i/\sqrt{2}\}$ ($X_0 = \mathbb{1}_d/\sqrt{d}$) is the orthonormal operator bases for $\mathcal{H}$ (e.g., X_i is the Pauli matrices when $d = 2$, and the Gell-Mann matrices when $d = 3$). Then if one arranges elements of $\vec{X}$ as

$$\vec{X} = \{u_{12}, v_{12}, \dots, u_{d-1,d}, v_{d-1,d}, w_1, \dots, w_{d-1}\}, \tag{155}$$

with the elements

$$u_{jk}=|j\rangle\langle k|+|k\rangle\langle j|,\ \ v_{jk}=-i(|j\rangle\langle k|-|k\rangle\langle j|),$$
$$w_l=\sqrt{\frac{2}{l(l+1)}}\sum_{j=1}^{l}(|j\rangle\langle j|-l|l+1\rangle\langle l+1|), \tag{156}$$

where $j,k\in\{1,2,\ldots,d\}$ with $j<k$, $l\in\{1,2,\ldots,d-1\}$ (the symbol i in v_{jk} is the imaginary unit), one has (Singh et al., 2015)

$$C_{l_1}(\rho)=\sum_{r=1}^{(d^2-d)/2}\sqrt{x_{2r-1}^2+x_{2r}^2}. \tag{157}$$

By using Eq. (157), Hu and Fan (2017) showed that the maximal $C_{l_1}(\rho)$ under generic basis is upper bounded by

$$C_{l_1}^{\max}(\rho)\le\sqrt{\frac{d^2-d}{2}}|\vec{x}|, \tag{158}$$

where $|\vec{x}|$ is length of the vector $(x_1,x_2,\ldots,x_{d^2-1})$.

3.1.3. Trace norm of coherence

Apart from the l_1 norm, one may wonder whether the general l_p and Schatten-p matrix norm can be adopted for defining coherence measures. In general, the answer to this question is negative. For example, Baumgratz et al. (2014) have considered the HS norm (i.e., $p=2$, for which it is also known as the Frobenius norm) measure of coherence defined as

$$C_{l_2}(\rho)=\min_{\delta\in\mathcal{I}}\|\rho-\delta\|_2^2=\sum_{i\neq j}|\langle i|\rho|j\rangle|^2, \tag{159}$$

and showed through a counterexample that it does not satisfy condition (C2b). Rana et al. (2016) further showed that any coherence measure defined via the l_p norm or the Schatten-p norm with $p\ge 2$ violates (C2b).

For the case of $p=1$ which corresponds to the trace norm (i.e., the Schatten-1 norm), if one defines

$$C_{tr}(\rho)=\min_{\delta\in\mathcal{I}}\|\rho-\delta\|_1, \tag{160}$$

with $\|M\|_1=\mathrm{tr}\sqrt{M^\dagger M}$ denoting the trace norm of the matrix M, the conditions (C1), (C2a), and (C3) for $C_{tr}(\rho)$ to be a proper coherence measure have been proven, but the work of Yu et al. (2016c) showed that the condition of Eq. (141) may be violated, thus proved it not to be a proper coherence measure.

For certain special classes of states, e.g., ρ of one qubit or having possible nonzero elements along only the main diagonal and anti-diagonal (i.e., the X states), $C_{tr}(\rho)$ has already been proven to be a coherence monotone, and the corresponding optimal incoherent state is given by ρ_{diag} (Bromley et al., 2015). Moreover, for the state ρ with all of its non-diagonal elements equal to each other, i.e., $\rho_{ij}=a$ ($\forall i\neq j$), the trace norm of coherence can be derived analytically as

$$C_{tr}(\rho)=2(d-1)|a|, \tag{161}$$

where $d=\dim\rho$, and the closest incoherent state is $\delta^\star=\rho_{\mathrm{diag}}$ (Wang et al., 2016c). For a restricted family of SIO (Winter and Yang, 2016), i.e., those of the SIO whose Kraus operators are $(2\times d)$-dimensional, the trace norm of coherence for this family of ρ was also showed to satisfy the four conditions for a reliable quantum coherence measure (Wang et al., 2016c). But its monotonicity under general IO does not hold.

When restricted to pure states $|\psi\rangle$, it is possible to identify structure of the optimal incoherent state under the trace norm of coherence. As for any pure state $|\psi\rangle$, one can find a diagonal unitary matrix U and a permutation matrix P which gives $PU|\psi\rangle=|x\rangle$, with entries $x_1\ge\cdots\ge x_d\ge 0$, the calculation can be performed to $|x\rangle$ only. By using the approximation theory, Chen et al. (2016b) found that $|\Psi_d\rangle$ of Eq. (137) is the unique state that maximizing the trace norm of coherence, for which $C_{tr}^{\max}=2(1-1/d)$. The optimal incoherent state to $|x\rangle$ and the corresponding trace norm of coherence are given, respectively, by

$$\delta_{\mathrm{opt}}=\mathrm{diag}\{\alpha_1,\ldots,\alpha_k,0,\ldots,0\},$$
$$C_{tr}(|x\rangle\langle x|)=2(q_ks_k+m_k), \tag{162}$$

where

$$\alpha_i=\frac{x_i-q_k}{s_k-kq_k}, \tag{163}$$

and k is the maximum integer satisfying

$$x_k>q_k:=\frac{1}{2ks_k}\left(p_k+\sqrt{p_k^2+4km_ks_k^2}\right), \tag{164}$$

with the parameters

$$s_l = \sum_{i=1}^{l} x_i,\ m_l = \sum_{i=l+1}^{d} x_i^2,\ p_l = s_l^2 - l m_l - 1 \tag{165}$$

for all $l \in \{1, 2, \ldots, d\}$.

To avoid the perplexity for $C_{tr}(\rho)$ of Eq. (160), i.e., the non-monotonicity of the trace norm of coherence under general incoherent operations, Yu et al. (2016c) further proposed a modified version of trace norm of coherence by introducing a control parameter λ, and defined it as

$$C'_{tr}(\rho) = \min_{\lambda\geq 0, \delta\in\mathcal{I}} \|\rho - \lambda\delta\|_1, \tag{166}$$

and proved that it satisfies the conditions (C1), (C2a) and Eq. (141), that is to say, it satisfies all the conditions for a reliable measure of quantum coherence. As its relation with other coherence measures, we have

$$C'_{tr}(\rho) \leq C_{tr}(\rho) \leq C_{l_1}(\rho), \tag{167}$$

where the first inequality is obvious from their definitions in Eqs. (160) and (166), and the second one is due to $\|\cdot\|_1 \leq \|\cdot\|_{l_1}$ for any Hermitian operator.

For one-qubit state, $C'_{tr}(\rho) = C_{tr}(\rho) = C_{l_1}(\rho)$, and the optimal parameter $\lambda^\star = 1$ and the optimal $\delta^\star = \rho_{\rm diag}$ (Bromley et al., 2015; Chen and Fei, 2018). For general state ρ, determination of the analytical solution of $C'_{tr}(\rho)$ is possible only for certain special family of states. For example, the class of maximally coherent mixed states (MCMS) with respect to the l_1 norm of coherence, up to incoherent unitaries, is given by (Singh et al., 2015)

$$\rho_{\rm mcms} = \frac{1-p}{d}\mathbb{1} + p|\Psi_d\rangle\langle\Psi_d|, \tag{168}$$

for which the modified trace norm of coherence can be obtained analytically as $C'_{tr}(\rho_{\rm mcms}) = p$, with the optimal $\lambda^\star = 1 - p$ and $\delta^\star = \mathbb{1}_d/d$.

3.2. Entanglement-based measure of coherence

In a way analogous to the entanglement activation via local von Neumann measurements (Streltsov et al., 2011b), one can also introduce the operational coherence measure with the help of IO.

Given a system S in the state ρ_S and an ancilla A initialized in the pure state $|0^A\rangle$, Streltsov et al. (2015) considered incoherent operations Λ^{SA} on the combined system SA. By denoting $E_\mathcal{D} = \min_{\chi\in\mathcal{S}} \mathcal{D}(\rho, \chi)$ a distance-based entanglement monotone and $C_\mathcal{D}$ the corresponding coherence monotone as given in Eq. (143), with $\mathcal{D}$ any contractive distance measure of quantum states and $\mathcal{S}$ the set of separable states, they found that the generated entanglement $E_\mathcal{D}^{S:A}$ is bounded from above by

$$E_\mathcal{D}^{S:A}(\Lambda^{SA}[\rho^S \otimes |0^A\rangle\langle 0^A|]) \leq C_\mathcal{D}(\rho^S), \tag{169}$$

which means that when ρ^S is incoherent, the IO cannot generate entanglement between S and A.

Particularly, when $\mathcal{D}$ is the relative entropy and $d_A \geq d_S$ with $d_{A,S} = \dim\mathcal{H}_{A,S}$, then there always exists an incoherent operation (i.e., the generalized CNOT operation)

$$U_{\rm CNOT} = \sum_{i=0}^{d_S-1}\sum_{j=0}^{d_S-1} |i, i\oplus j\rangle^{SA}\langle ij| + \sum_{i=0}^{d_S-1}\sum_{j=d_S}^{d_A-1} |ij\rangle^{SA}\langle ij|, \tag{170}$$

where $\oplus$ represents addition modulo d_S. This unitary operation maps the state $\rho^S \otimes |0^A\rangle\langle 0^A|$ to

$$\Lambda^{SA}[\rho^S \otimes |0^A\rangle\langle 0^A|] = \sum_{ij} \rho_{ij}^S |ij\rangle^{SA}\langle ij|, \tag{171}$$

and henceforth Eq. (169) is saturated:

$$E_r^{S:A}(\Lambda^{SA}[\rho^S \otimes |0^A\rangle\langle 0^A|]) = C_r(\rho^S), \tag{172}$$

which can be proved immediately by Eq. (169) and the lower bound $-S(A|S)$ of $E_r^{S:A}(\rho^{SA})$, where $S(A|S)$ is the conditional entropy. Then, Streltsov et al. (2015) proposed to define coherence of ρ^S as the maximal entanglement of SA generated by IO, that is,

$$C_E(\rho^S) = \lim_{d_A\to\infty} \{\sup_{\Lambda^{SA}} E^{S:A}(\Lambda^{SA}[\rho^S \otimes |0^A\rangle\langle 0^A|])\}, \tag{173}$$

with E being an arbitrary entanglement measure, and C_E will satisfy the four conditions of Baumgratz et al. (2014) if E is convex as well.

For the geometric measure of entanglement $E_g(\rho) = 1 - \max_{\sigma\in\mathcal{S}} F(\rho,\sigma)$ [see also Eq. (12)] (Streltsov et al., 2010a), the associated coherence measure can be evaluated as

$$C_g(\rho) = 1 - \max_{\delta\in\mathcal{I}} F(\rho,\delta), \tag{174}$$

and for ρ of single-qubit state,

$$C_g(\rho) = \frac{1}{2}\left(1 - \sqrt{1-4|\rho_{12}|^2}\right), \tag{175}$$

which is an increasing function of $C_{l_1}(\rho) = 2|\rho_{12}|$.

Moreover, for pure state $\psi = |\psi\rangle\langle\psi|$, as $F(\psi,\sigma) = |\langle\psi|\sigma|\psi\rangle|$, we have

$$C_g(\psi) = 1 - \max_i\{\psi_{ii}\}, \tag{176}$$

with ψ_{ii} being the diagonal elements of ψ. For general mixed states, the calculation of $C_g(\rho)$ is formidably difficult, hence derives some bounds of it is significant. By using the relations among fidelity $F(\rho,\sigma)$, sub-fidelity $E(\rho,\sigma)$, and super-fidelity $G(\rho,\sigma)$, Zhang et al. (2017a) obtained lower and upper bounds of $C_g(\rho)$. The sub-fidelity and super-fidelity were defined as (Miszczak et al., 2009)

$$\begin{aligned} E(\rho,\sigma) &= \mathrm{tr}(\rho\sigma) + \sqrt{2[(\mathrm{tr}(\rho\sigma))^2 - \mathrm{tr}(\rho\sigma\rho\sigma)]}, \\ G(\rho,\sigma) &= \mathrm{tr}(\rho\sigma) + \sqrt{(1-\mathrm{tr}\rho^2)(1-\mathrm{tr}\sigma^2)}, \end{aligned} \tag{177}$$

and based on the relation $E(\rho,\sigma) \le F(\rho,\sigma) \le G(\rho,\sigma)$ (the equality holds for one-qubit state or at least one of ρ and σ is pure), they found

$$\begin{aligned} &1 - \frac{1}{d} - \frac{d-1}{d}\sqrt{1 - \frac{d}{d-1}\left(\mathrm{tr}\rho^2 - \sum_i \rho_{ii}^2\right)} \\ &\le C_g(\rho) \le \min\left\{1 - \max_i\{\rho_{ii}\},\, 1 - \sum_i b_{ii}^2\right\}, \end{aligned} \tag{178}$$

where b_{ii} is related to the square root of $\sqrt{\rho} = \sum_{ij} b_{ij}|i\rangle\langle j|$.

3.3. Convex roof measure of coherence

3.3.1. Intrinsic randomness of coherence

In the framework of quantum theory, measurement of quantum states induce intrinsically random outputs in general, and this randomness indicates genuine quantumness of a system. Based on this consideration, Yuan et al. (2015) proposed a convex roof measure for coherence, which has been proved to satisfy the four conditions of Baumgratz et al. (2014). We call it intrinsic randomness of coherence.

For pure states $\psi = |\psi\rangle\langle\psi|$, the intrinsic randomness can be quantified by Shannon entropy of the probability distribution $\{p_i\}$ of the measurement outcomes which reads

$$R_I(\psi) = H(\{p_i\}) = -\sum_i p_i \log_2 p_i, \tag{179}$$

with $H(\{p_i\})$ denotes the Shannon entropy of the probability distribution $\{p_i\}$, with $p_i = \mathrm{tr}(E_i\psi)$ and $\{E_i\}$ is the set of measurement operators. R_I characterizes also the degree of uncertainty related to the measurement outcomes, namely, the outcomes that cannot be predicted by blindly guessing. When restricted to projective measurements for which $E_i = |i\rangle\langle i|$ with $\{|i\rangle\}$ the reference basis, the right-hand side of Eq. (179) is $S(\psi_{\mathrm{diag}})$, henceforth, the intrinsic randomness $R_I(\psi)$ equals the relative entropy of coherence $C_r(\psi)$.

For general case of mixed states ρ, Yuan et al. (2015) defined the intrinsic randomness $R_I(\rho)$ by utilizing convex roof construction, that is,

$$R_I(\rho) = \min_{\{p_i,\psi_i\}} \sum_i p_i R_I(\psi_i), \tag{180}$$

where $\sum_i p_i = 1$, $\psi_i = |\psi_i\rangle\langle\psi_i|$, and the minimum is taken over all possible pure state decompositions of ρ given in Eq. (13).

Eq. (180) establishes a convex roof definition of quantum coherence, which bears some resemblance with the convex roof measures of entanglement such as entanglement of formation $E_f(\rho) = \min_{\{p_i,\psi_i\}} \sum_i p_i S(\mathrm{tr}_B\psi_i)$ and the geometric entanglement $E_g(\rho) = \min_{\{p_i,\psi_i\}}\sum_i p_i E_g(\psi_i)$. It was also termed as superposition of formation by Aberg (2006), and coherence of formation by Winter and Yang (2016).

Apart from pure states, $R_I(\rho)$ is analytically computable for one-qubit states, i.e.,

$$R_I(\rho) = H\left(\frac{1+\sqrt{1-C_z^2(\rho)}}{2}\right), \tag{181}$$

where $C_z(\rho) = 2|\rho_{12}|$ is termed as coherence concurrence, as it is given by

$$C_z(\rho) = |\lambda_1 - \lambda_2|, \tag{182}$$

with λ_i being square roots of the eigenvalues of the product matrix

$$R = \rho\sigma_x\rho^*\sigma_x, \tag{183}$$

where ρ^* is the conjugation of ρ, and σ_x is the first Pauli matrix.

3.3.2. Coherence concurrence

Using the fact that any ρ can be decomposed as Eq. (154), Qi et al. (2017) found that the l_1 norm of coherence for ρ is equivalent to

$$C_{l_1}(\rho) = \sum_{j<k}\left|\sqrt{\eta_1^{jk}} - \sqrt{\eta_2^{jk}}\right|, \tag{184}$$

where η_1^{jk} and η_2^{jk} are nonzero eigenvalues of the matrix

$$R^{jk} = \rho u_{jk}\rho^* u_{jk}, \tag{185}$$

and the generalized Gell-Mann matrices $\{u_{jk}\}$ as given in Eq. (156). When $d = 2$, R^{jk} is just that of R given in Eq. (183). For pure state $\psi = |\psi\rangle\langle\psi|$, the above equation further gives

$$C_{l_1}(\psi) = \sum_{j<k}|\langle\psi|u_{jk}|\psi^*\rangle|, \tag{186}$$

which is direct consequence of $R^{jk} = |\psi\rangle\langle\psi|u_{jk}|\psi^*\rangle\langle\psi^*|u_{jk}$. Motivated by this fact, Qi et al. (2017) proposed a convex roof measure of coherence

$$C_{\rm con}(\rho) = \min_{\{p_i,\psi_i\}}\sum_i p_i C_{l_1}(\psi_i), \tag{187}$$

where the minimization is with respect to the possible pure state decompositions of ρ given in Eq. (13). $C_{\rm con}(\rho)$ is termed as coherence concurrence, as it is very similar to the entanglement concurrence given by

$$C_E(\rho) = \min_{\{p_i,\psi_i\}}\sum_i p_i C_E(\psi_i), \tag{188}$$

with $C_E(\psi_i) = [2(1-{\rm tr}\rho_A^2)]^{1/2}$, and $\rho_A = {\rm tr}_B\psi_i$ is the reduced density matrix of ψ_i. For two-qubit state, $C_E(\rho)$ is analytically solved as $C_E(\rho) = \max\{0, 2\lambda_{\max} - \sum_{j=1}^4\lambda_j\}$ (with λ_js being eigenvalues of $\rho(\sigma_y\otimes\sigma_y)\rho^*(\sigma_y\otimes\sigma_y)$), and it is linked to the entanglement of formation as $E_f = H([1+(1-C_E^2)^{1/2}]/2)$, with $H(\cdot)$ being the binary Shannon entropy function (Bennett et al., 1996; Wootters, 1998).

It can be proved that this measure satisfies all the conditions (C1), (C2a), (C2b), and (C3) for a reliable measure of quantum coherence. Moreover, it is bounded from below by the l_1 norm of coherence, i.e., $C_{\rm con}(\rho) \geq C_{l_1}(\rho)$, which can be derived directly by combining the definition (187) and the convexity of it.

3.3.3. Fidelity-based measure of coherence

Liu et al. (2017b) proposed a convex roof measure of coherence based on fidelity and showed that it fulfills the four conditions introduced by Baumgratz et al. (2014). For any pure state $\psi = |\psi\rangle\langle\psi|$, it is defined as

$$C_F(\psi) = \min_{\delta\in\mathcal{I}}\sqrt{1-F(\psi,\delta)}, \tag{189}$$

which is very similar to $C_g(\rho)$ of Eq. (174), and can be derived analytically as $C_F(\psi) = C_g^{1/2}(\psi)$ [see Eq. (176)]. For general mixed state ρ, it is defined via a convex roof construction, that is

$$C_F(\rho) = \min_{\{p_i,\psi_i\}}\sum_i p_i C_F(\psi_i), \tag{190}$$

and the minimization is taken over all the possible pure state decompositions of ρ, see Eq. (13). Moreover, when ρ is of a single-qubit state, $C_F(\rho) = C_g^{1/2}(\rho)$, see Eq. (175).

3.3.4. The rank-measure of coherence

The Schmidt rank for a pure state and the Schmidt number which is an extension of the Schmidt rank to mixed states have been shown to be useful for defining entanglement measures (Gour, 2005). The Schmidt rank $r(\psi)$ for a $(d\times d')$-dimensional pure state $\psi = |\psi\rangle\langle\psi|$ is the number of nonzero coefficients in its Schmidt decomposition of Eq. (11), and the Schmidt number for a general $(d\times d')$-dimensional mixed state ρ_{AB} is defined as (Terhal and Horodecki, 2000)

$$r(\rho_{AB}) = \min_{\{p_i,\psi_i\}} \max_i r(\psi_i), \tag{191}$$

where $\psi_i = |\psi_i\rangle\langle\psi_i|$, and the minimization is taken over all the pure state decompositions of ρ, see Eq. (13).

In analogous to Schmidt rank and Schmidt number, the coherence rank $r_C(\psi) = \mathrm{rank}(\psi)$ is the number of nonzero coefficients α_i for a pure state $|\psi\rangle = \sum_i \alpha_i |i\rangle$. It can serve as a coherence measure of ψ. For a general mixed state ρ, Chin (2017) introduced a convex roof measure of coherence termed as coherence number. It reads

$$r_C(\rho) = \min_{\{p_i,\psi_i\}} \max_i r_C(\psi_i), \tag{192}$$

where the minimization is taken over the pure state decompositions of ρ showed in Eq. (13). This is a coherence monotone under IO, as it satisfies the conditions (C1), (C2a), and (C2b) of Baumgratz et al. (2014).

It is obvious that $r_C(|\Psi_d\rangle) = d$ for the maximally incoherent state $|\Psi_d\rangle$, and $r_C(\delta) = 1$ for any incoherent state δ. Of course, one can also take logarithm to $r_C(\rho)$ and define the coherence monotone as (Zhao et al., 2018)

$$C_0(\rho) = \log_2 r_C(\rho), \tag{193}$$

and now $C_0(\delta)$ equals zero when δ is incoherent.

3.4. Robustness of coherence

Given an arbitrary state ρ on the Hilbert space $\mathcal{H}$, its convex mixture with another state τ on the same space may be coherent or incoherent. In another word, a proper choice of τ and weight factor s of mixing may destroy the coherence in ρ. Based on this fact and stimulated by similar definitions for various quantum correlation monotones, Napoli et al. (2016) introduced a new coherence measure which was called robustness of coherence (RoC). It is defined as

$$C_R(\rho) = \min_{\tau\in\mathcal{D}(\mathbb{C}^d)} \left\{ s\geq 0 \,\middle|\, \frac{\rho+s\tau}{1+s} := \delta\in\mathcal{I} \right\}, \tag{194}$$

where $\mathcal{D}(\mathbb{C}^d)$ is the convex set of density operators on $\mathcal{H}$.

Alternatively, the RoC can also be defined as (Piani et al., 2016)

$$C_R(\rho) = \min_{\delta\in\mathcal{I}} \{s\geq 0 | \rho \leq (1+s)\delta\}. \tag{195}$$

The RoC is proved to be a full coherence monotone (Napoli et al., 2016). That is to say, it satisfies the conditions required by the framework for a resource theory of quantum coherence (Baumgratz et al., 2014). It is also analytically computable for ρ being arbitrary one-qubit and pure states, as well as for those ρ with possible nonzero elements along only the main diagonal and anti-diagonal (i.e., the so-called X states). For those ρ, $C_R(\rho) = \sum_{i\neq j}|\rho_{ij}|$ in the reference basis $\{|i\rangle\}$, hence equals the related l_1 norm of coherence $C_{l_1}(\rho)$.

For general ρ, Napoli et al. (2016) constructed a semidefinite program for calculating $C_R(\rho)$ numerically, and obtained tight lower bounds of RoC of the following form (Piani et al., 2016)

$$C_R(\rho) \geq \frac{\|\rho-\Delta(\rho)\|_2^2}{\|\Delta(\rho)\|_\infty} \geq \frac{\|\rho-\Delta(\rho)\|_2^2}{\|\Delta(\rho)\|_2} \geq \|\rho-\Delta(\rho)\|_2^2, \tag{196}$$

where the operator norm $\|M\|_\infty = \lambda_{\max}$, with $\lambda_{\max}$ being the largest singular value of M, and $\|M\|_2$ is the HS norm.

Napoli et al. (2016) also obtained bounds of $C_R(\rho)$ via the l_1 norm of coherence which is analytically computable, i.e.,

$$(d-1)^{-1} C_{l_1}(\rho) \leq C_R(\rho) \leq C_{l_1}(\rho), \tag{197}$$

where the upper bound is tight obviously, and the lower bound is saturated by the family of $\rho = (1+p)\mathbb{1}/d - p|\Psi_d\rangle\langle\Psi_d|$, with $0\leq p\leq 1/(d-1)$. It has also been proven that (Rana et al., 2017)

$$C_r(\rho) \leq \log_2[1+C_R(\rho)]. \tag{198}$$

Moreover, the RoC is also shown to be upper bounded by (Piani et al., 2016)

$$C_R(\rho) \leq d\|\rho\|_\infty - 1, \tag{199}$$

where $\|\rho\|_\infty$ denotes the largest singular value of ρ (also known as the operator norm). This inequality implies that $C_R(\rho)$ takes its maximum value $d-1$ only if ρ is a maximally coherent pure state.

From Eq. (156) one can verify directly that the optimization in Eq. (194) can be restricted to the subset of $\tau \in \mathcal{D}(\mathbb{C}^d)$ given by

$$\tau_{\rm sub} = \frac{1}{d}\mathbb{1}_d - \frac{1}{2}\sum_{i=1}^{d_0} x_i X_i + \frac{1}{2}\sum_{i=d_0+1}^{d^2-1} y_i X_i, \tag{200}$$

where $d_0 = (d^2-d)/2$. That is to say, the optimization can be performed over all possible $\{y_i\}$ such that $\tau_{\rm sub}$ is a physically allowed state.

Experimentally, Wang et al. (2017) have explored the RoC for one-qubit states. They developed two different methods to measure directly the quantum coherence, i.e., the interference-fringe method and the witness-observable method. For the former one, they showed experimentally that the sweeping on ancilla state is necessary only along the equatorial pure states, while for the latter one, the optimal witness operator is

$$W^* = \cos\varphi_\rho \sigma_1 + \sin\varphi_\rho \sigma_2. \tag{201}$$

They have also compared the experimental results with those calculated via state tomography and found a high coincidence of them.

Stimulated by the resource weight-based quantification of quantum features such as the best separable approximation of entangled states (Lewenstein and Sanpera, 1998), the steering weight (Skrzypczyk et al., 2014), and the measurement incompatibility weight (Pusey, 2015), Bu et al. (2018) proposed a similar measure of coherence which they termed as coherence weight. It is defined as

$$C_w(\rho) = \min_{\delta\in\mathcal{I},\tau\in\mathcal{D}(\mathbb{C}^d)} \{s \geq 0 | \rho = (1-s)\delta + s\tau\}, \tag{202}$$

which can be interpreted operationally as the minimal number of coherent states (combination with the maximal number of incoherent states) needed to prepare the considered state ρ on average. $C_w(\rho)$ was shown to satisfy all the four conditions proposed by Baumgratz et al. (2014), thus constitutes a bona fide measure of coherence. For certain special states, the coherence weight can be obtained analytically, e.g., for the pure states we always have $C_w(\rho) = 1$, while for Werner state ρ_W of Eq. (25), we have

$$C_w(\rho_W) = C_R(\rho_W) = C_{l_1}(\rho_W) = \frac{1-dx}{d+1}. \tag{203}$$

Similar to the definition of RoC in Eq. (195), the coherence weight can also be defined as

$$C_w(\rho) = \min_{\delta\in\mathcal{I}} \{s \geq 0 | \rho \geq (1-s)\delta\}. \tag{204}$$

Bu et al. (2018) also obtained lower bounds of $C_w(\rho)$ as

$$C_w(\rho) \geq \frac{\|\rho - \Delta(\rho)\|_2^2}{\|\rho\|_\infty} \geq \|\rho - \Delta(\rho)\|_2^2, \tag{205}$$

and proved its relation with the other quantum coherence measures, i.e.,

$$\begin{aligned} &C_w(\rho) \geq \frac{1}{d-1} C_{l_1}(\rho) \geq \frac{1}{d-1} C_R(\rho), \\ &C_w(\rho) \geq \frac{1}{\ln d} C_r(\rho). \end{aligned} \tag{206}$$

Moreover, the coherence weight C_w is shown to satisfy the following relation

$$C_w(\rho_1 \otimes \rho_2) \leq C_w(\rho_1) + C_w(\rho_2) - C_w(\rho_1)C_w(\rho_2), \tag{207}$$

for any quantum states ρ_1 and ρ_2, while for RoC, we have

$$C_R(\rho_1 \otimes \rho_2) \leq C_R(\rho_1) + C_R(\rho_2) + C_R(\rho_1)C_R(\rho_2). \tag{208}$$

3.5. Tsallis relative entropy measure of coherence

Rastegin (2016) studied the problem for quantifying coherence via the Tsallis α relative entropies, which are functionals of powers of density matrices, and formulated family of coherence measures defined by quantum divergences of the Tsallis type.

As an extension of the standard quantum relative entropy, the Tsallis α divergence is given by

$$D_\alpha(\rho \parallel \sigma) = \begin{cases} \dfrac{\mathrm{tr}(\rho^\alpha \sigma^{1-\alpha}) - 1}{\alpha - 1}, & \text{if } \mathrm{ran}(\rho) \subseteq \mathrm{ran}(\sigma), \\ +\infty, & \text{otherwise,} \end{cases} \tag{209}$$

with α being a positive number. The trace is taken over $\mathrm{ran}(\sigma)$ for $\alpha > 1$, while for $\alpha \in (0, 1)$ this condition is not necessary. Here, $\mathrm{ran}(\rho)$ denotes the range of ρ, and likewise for $\mathrm{ran}(\sigma)$.

Then, motivated by Eq. (143), Rastegin (2016) proposed to quantify coherence of ρ as

$$C_\alpha(\rho) = \min_{\delta\in\mathcal{I}} D_\alpha(\rho \parallel \delta), \tag{210}$$

and further proved that it can be evaluated analytically as

$$C_\alpha(\rho) = \frac{1}{\alpha-1}\left\{\left(\sum_i \langle i|\rho^\alpha|i\rangle^\alpha\right)^{1/\alpha} - 1\right\}, \tag{211}$$

and for $\alpha = 1$, it reduces to the relative entropy of coherence in Eq. (144), while for the specific case of $\alpha = 2$, we have

$$C_2(\rho) = \left(\sum_j \sqrt{\sum_i |\langle i|\rho|j\rangle|^2}\right)^2 - 1, \tag{212}$$

The coherence measure based on the quantum α divergence is bounded above by

$$C_\alpha(\rho) \leq \begin{cases} -\ln_\alpha \dfrac{1}{d\mathrm{tr}\rho^2}, & \text{if } \alpha \in (0, 2], \\ \dfrac{d\mathrm{tr}\rho^2 \varsigma^{\alpha-2} - 1}{\alpha - 1}, & \text{if } \alpha \in (2, +\infty), \end{cases} \tag{213}$$

where $\varsigma = \{(d-1)(d\mathrm{tr}\rho^2 - 1)\}^{1/2} + 1$, which is intimately related to the mixedness of ρ, hence is experimentally accessible.

To be a reliable quantifier of coherence, the proposed functional should obey the four conditions derived via the resource theoretic framework of coherence (Baumgratz et al., 2014). For $C_\alpha(\rho)$, it vanishes if and only if $\rho \in \mathcal{I}$, namely, it satisfies the condition (C1). Furthermore, (C2a) is satisfied for $\alpha \in (0, 2]$, which is a direct result of $D_\alpha(\Lambda[\rho], \Lambda[\sigma]) \leq D_\alpha(\rho, \sigma)$ for the Tsallis α divergence. Thirdly, by denoting δ^* the closest incoherent state to ρ, and $q_i = \mathrm{tr}(K_i\delta^* K_i^\dagger)$, $p_i = \mathrm{tr}(K_i\rho K_i^\dagger)$, and $\rho_i = E_i\rho E_i^\dagger/p_i$, Rastegin (2016) derived a generalized form of the monotonicity formulation applicable for $C_\alpha(\rho)$. It is given by

$$C_\alpha(\rho) \geq \sum_i p_i^\alpha q_i^{1-\alpha} C_\alpha(\rho_i), \tag{214}$$

which reduces to the usual monotonicity formula for standard relative entropy when $\alpha = 1$. Note that the coherence measure of Eq. (209) may violate the monotonicity condition (C2b) for $\alpha \neq 1$. Finally, the Tsallis α relative entropy measure of coherence is also convex for $\alpha \in (0, 2]$, this is because the Tsallis divergence D_α is a convex function of density matrices in the same region of α.

If one takes logarithm (with base 2) to the first term of Eq. (209), then in the limit of $\alpha \to \infty$, we can recover the maximum relative entropy defined as

$$D_{\max}(\rho \parallel \sigma) = \min\{\lambda | \rho \leq 2^\lambda \sigma\}, \tag{215}$$

where $\lambda \geq 0$. It is also an important concept in quantum information science (Datta, 2009b, c; Buscemi and Datta, 2010; Brandão and Datta, 2011). In a recent work, Bu et al. (2017b) and Zhao et al. (2018) proposed to use it as a basis for defining a coherence measure which was termed as the maximum relative entropy of coherence. It reads

$$C_{\max}(\rho) := \min_{\delta\in\mathcal{I}} D_{\max}(\rho \parallel \delta), \tag{216}$$

and has been proven to obey the conditions (C1), (C2a), and C(2b) introduced by Baumgratz et al. (2014), so it is a coherence monotone under MIO (Zhao et al., 2018). Nevertheless, it does not satisfy the convexity condition (C3), and it is only quasiconvex, that is, $C_{\max}(\sum_i p_i\rho_i) \leq \max_i C_{\max}(\rho_i)$. Moreover, by comparing the above equation with Eq. (195), one can also found that $C_{\max}(\rho)$ can be linked quantitatively to the RoC as

$$C_{\max}(\rho) = \log_2[1 + C_R(\rho)], \tag{217}$$

thus similar to RoC, if $\exists U$ such that $(U\rho U^\dagger)_{kl} = |\rho_{kl}|$, then a closed formula for $C_{\max}(\rho)$ can also be obtained. A class of ρ where such a requirement is satisfied consists of all the one-qubit states, the pure states, and the X states.

Besides $C_{\max}(\rho)$, Bu et al. (2017b) further proposed the ε-smoothed maximum relative entropy of coherence, which was defined as

$$C_{\max}^\varepsilon(\rho) := \min_{\rho'\in B_\varepsilon(\rho)} C_{\max}(\rho'), \tag{218}$$

where $B_\varepsilon(\rho) := \{\rho' \geq 0\colon \|\rho' - \rho\|_1 \leq \varepsilon, \operatorname{tr}\rho' \leq \operatorname{tr}\rho\}$. It has been shown that $\lim_{\varepsilon\to 0, n\to\infty} C^{\varepsilon}_{\max}(\rho^{\otimes n}) = nC_r(\rho)$. That is, $C^{\varepsilon}_{\max}(\rho)$ is equivalent to the relative entropy of coherence in the asymptotic limit.

Chitambar and Gour (2016b) put forward another similar coherence measure as

$$C_{\Delta,\max}(\rho) := \min\{\lambda | \rho \leq 2^{\lambda} \Delta(\rho)\}, \tag{219}$$

which has also been proven to be a coherence monotone under DIO (Chitambar and Gour, 2016b; Zhao et al., 2018) and it is also quasiconvex.

3.6. Skew-information-based measure of coherence

Soon after the work of Baumgratz et al. (2014), Girolami (2014) proposed a new method to quantify the amount of coherence in a state. It is defined based on the WY skew information, and has later been proven to violate one of the reliability criteria for a bona fide coherence monotone (Du and Bai, 2015). But due to the experimental accessibility, it may still be of interest to the quantum community.

3.6.1. Definition and properties

For a system to be measured, the uncertainty of the outputs comes from both ignorance of the mixture of the state and the truly quantum part related to state collapse induced by measurements. Girolami (2014) proposed that the latter feature (i.e., the truly quantum uncertainty of a measurement) is an embodiment of quantum coherence, and can be reliably quantified by the WY skew information.

The skew information $\mathsf{I}(\rho, K)$ is nonnegative and vanishes if and only if $[\rho, K] = 0$, namely, if and only if ρ is diagonal in the basis defined by K, hence it fulfills condition (C1). Furthermore, $\mathsf{I}(\rho, K)$ is a convex function of density matrices, thus (C3) is also fulfilled. But $\mathsf{I}(\rho, K)$ does not fulfill the other axiomatic postulates for a faithful coherence measure (Baumgratz et al., 2014), e.g., Du and Bai (2015) have constructed a series of phase sensitive IO Λ for which $\mathsf{I}(\rho, K) \leq \mathsf{I}(\Lambda[\rho], K)$.

But the above fact does not affect the status of $\mathsf{I}(\rho, K)$ as a well-accepted measure of asymmetry, which is defined with respect to a given symmetry group G, and includes as a special case the group U(1) used for defining quantum coherence. In fact, previously Marvian and Spekkens (2014) have proposed such an asymmetry measure, where they used the more general $\mathsf{I}^p(\rho, L)$ of Eq. (75) with $p \in (0, 1) \cup (1, 2]$, i.e.,

$$S_{L,p}(\rho) = \operatorname{tr}(\rho L^2) - \operatorname{tr}(\rho^p L \rho^{1-p} L), \tag{220}$$

and L represents an arbitrary generator of the group. If ρ is symmetric relative to group G $[U_g(\rho) = \rho$ for all group elements $g \in \mathrm{G}]$, then $S_{L,p}(\rho) = 0$, see Marvian and Spekkens (2014) and Marvian (2012). In the same work, the authors also proposed several other measures of asymmetry, for example, those based on the Holevo quantity, the trace norm, and the relative Rényi entropy [see Eq. (209)]. Moreover, Piani et al. (2016) also proposed a measure of asymmetry which they called robustness of asymmetry. It is defined in a manner very similar to RoC in Eq. (194), with only δ being replaced by the symmetric state relative to a group G.

In a more general sense, all the coherence measures defined in the framework of Baumgratz et al. (2014) constitute a proper subset of measures of asymmetry. Asymmetry measures the extent to which the symmetry relative to a group of translations such as time translations or phase shifts is broken, wherein the translationally invariant operations Λ_{TI} play a central role. To be explicit, an asymmetry measure f from states to reals should satisfy the inequality (Marvian and Spekkens, 2016)

$$f(\Lambda_{\mathrm{TI}}[\rho]) \leq f(\rho). \tag{221}$$

This answers why the WY skew information which measures asymmetry relative to the group of translations generated by an observable H cannot serve as a measure of coherence, as there are IO which are not translationally invariant. For more detailed explanations about the relations between coherence and asymmetry, and a comparison of different notions of coherence, see the recent works of Marvian and Spekkens (2016) and Marvian et al. (2016).

3.6.2. Tight lower bounds

By adopting the inequality

$$\operatorname{tr}\{[\rho, K]^2\} \geq 2\operatorname{tr}\{[\rho^{1/2}, K]^2\}, \tag{222}$$

Girolami (2014) derived the following lower bound of $\mathsf{I}(\rho, K)$,

$$\mathsf{I}(\rho, K) \geq \mathsf{I}^L(\rho, K) = -\frac{1}{4}\operatorname{tr}\{[\rho, K]^2\}, \tag{223}$$

and demonstrated that it can be experimentally evaluated efficiently without tomographic state reconstruction of the full density matrix.

Please cite this article in press as: Hu M.-L., et al., Quantum coherence and geometric quantum discord. Physics Reports (2018), https://doi.org/10.1016/j.physrep.2018.07.004.

Pires et al. (2015) also established a lower bound for the skew information. By discussing a dynamical process with the evolved state $\rho_\phi = U_\phi \rho_0 U_\phi^\dagger$ and the observable K_ϕ generating its evolution, they derived

$$\mathrm{I}(\rho_\phi, K_\phi) \geq \frac{\hbar^2}{2}\left|\frac{\partial}{\partial\phi}\cos[\mathcal{L}(\rho_0, \rho_\phi)]\right|^2, \tag{224}$$

where $\mathcal{L}(\rho_0, \rho_\phi) = \arccos[\mathrm{tr}(\rho_0^{1/2}\rho_\phi^{1/2})]$ is the Hellinger angle, and K_ϕ is connected to the unitary operator U_ϕ via

$$K_\phi = -i\hbar U_\phi \frac{\partial U_\phi^\dagger}{\partial\phi}, \tag{225}$$

with ϕ being an arbitrary parameter encoded in U_ϕ. Eq. (224) indicates that the lower bound of the evolved WY skew information $\mathrm{I}(\rho_\phi, K_\phi)$ is determined by change rate of the distinguishability between the initial and the evolved states.

3.6.3. *Modified version of coherence measure*

To avoid the problem occurred for $\mathrm{I}(\rho, K)$, Yu (2017) further proposed a similar definition of coherence measure still by using the WY skew information, which is very similar to the definition of quantum correlation measure $Q_A(\rho)$ given in Eq. (86). To be explicit, by denoting $\{|k\rangle\}$ the reference basis, the new coherence measure is defined as

$$C_{sk}(\rho) = \sum_k \mathrm{I}(\rho, |k\rangle\langle k|). \tag{226}$$

which has been proven to satisfy all the conditions for a bona fide measure of quantum coherence (Baumgratz et al., 2014). It can be linked to the task of quantum phase estimation. For the special case of single-qubit state, $C_{sk}(\rho)$ is also qualitatively equivalent to the asymmetry measure $\mathrm{I}(\rho, K)$ given by Girolami (2014).

To calculate $C_{sk}(\rho)$ for a given state ρ, one can also use its equivalent form

$$C_{sk}(\rho) = 1 - \sum_k \langle k|\sqrt{\rho}|k\rangle^2, \tag{227}$$

which can be further written in a distance-based form $C_{sk}(\rho) = 1 - [\max_{\delta\in\mathcal{I}} f(\rho, \delta)]^2$, where $f(\rho, \delta) = \mathrm{tr}(\sqrt{\rho}\sqrt{\delta})$. The optimal $\delta^\star$ can be derived as

$$\delta^\star = \sum_k \frac{\langle k|\sqrt{\rho}|k\rangle^2}{\sum_{k'}\langle k'|\sqrt{\rho}|k'\rangle^2}. \tag{228}$$

Moreover, by using Eq. (223) and the inequality $\langle k|\sqrt{\rho}|k\rangle \geq \langle k|\rho|k\rangle$, a connection between $C_{sk}(\rho)$ and the HS norm of coherence measure $C_{l_2}(\rho)$ given in Eq. (159) can be established as follows

$$\frac{1}{2}C_{l_2}(\rho) \leq C_{sk}(\rho) \leq 1 - \mathrm{tr}\rho^2 + C_{l_2}(\rho). \tag{229}$$

Since $C_{l_2}(\rho)$ is experimentally measurable, the above relation provides a way for estimating bounds of $C_{sk}(\rho)$.

3.7. *Coherence of Gaussian states*

In real experiments, there exist very relevant physical situations for which the systems under scrutiny are of infinite-dimensional (e.g., quantum optics states of light and Gaussian states). Hence, the characterization and quantification of coherence in these systems are also required.

3.7.1. *Coherence in the Fock space*

A typical class of infinite-dimensional systems is the bosonic system in the Fock space, which is describable using the Fock basis $\{|n\rangle\}_{n=0}^\infty$. Here, $|n\rangle$ is the eigenstate of the number operator $\hat{a}^\dagger\hat{a}$, and $\hat{a}^\dagger$ and $\hat{a}$ are the bosonic creation and annihilation operators.

By generalizing the set $\mathcal{I}$ of incoherent states as those with $\delta = \sum_{n=0}^\infty \delta_n |n\rangle\langle n|$, and incoherent operations described by the Kraus operators $\{K_n\}$ satisfying $\sum_{n=0}^\infty K_n^\dagger K_n = \mathbb{1}$ and $K_n \mathcal{I} K_n^\dagger \subset \mathcal{I}$, Zhang et al. (2016) studied the problem of quantification of coherence in this system. For this purpose, they first proposed a new criterion that $C(\rho)$ should satisfy in order to circumvent the divergence problem of $C(\rho)$, which may be termed as the mean energy constraints.

(C5) If the first-order moment, the average particle number $\bar{n} = \langle\hat{a}^\dagger\hat{a}\rangle$ is finite, it should fulfill $C(\rho) < \infty$.

Based on this new criterion, Zhang et al. (2016) proved that the relative entropy of coherence in Eq. (144) is also a proper coherence measure for the infinite-dimensional systems. But the l_1 norm of coherence of Eq. (147), despite its simple structure and intuitive meaning, does not satisfy the condition (C5), hence fails to serve as a proper measure of coherence for the infinite-dimensional systems.

Referring to the coherence measure in infinite-dimensional systems, one may also concern about the counterparts of Eqs. (136) and (137), i.e., the incoherent state and the maximally coherent state. For the d-mode Fock space $\mathcal{H}=\otimes_{i=1}^{d}\mathcal{H}_i$ with the basis $|\boldsymbol{n}\rangle=\otimes_{i=1}^{d}|n_i\rangle$ and probability distributions $\{p_{\boldsymbol{n}}\}$, the incoherent state is given by (Xu, 2016)

$$\delta_{\rm th}(\boldsymbol{n})=\otimes_{i=1}^{d}\rho_{\rm th}(\bar{n}_i), \tag{230}$$

where $\bar{n}_i=\langle\hat{a}_i^\dagger\hat{a}_i\rangle$, and $\rho_{\rm th}(\bar{n}_i)$ is just the thermal state for the ith-mode Fock space,

$$\rho_{\rm th}(\bar{n}_i)=\sum_{n=0}^{\infty}\frac{\bar{n}_i^n}{(\bar{n}_i+1)^{n+1}}|n\rangle\langle n|. \tag{231}$$

Moreover, the maximally coherent state is given by (Zhang et al., 2016)

$$|\Psi_m^d\rangle=\sum_{\boldsymbol{n}}\frac{\bar{n}_t^{|\boldsymbol{n}|_1/2}}{[(\bar{n}_t+1)^{|\boldsymbol{n}|_1+1}\mathbb{C}_{|\boldsymbol{n}|_1+d-1}^{d-1}]^{1/2}}|\boldsymbol{n}\rangle, \tag{232}$$

where $|\boldsymbol{n}|_1=\sum_{i=1}^{d}n_i$, and $\bar{n}_t=\sum_{\boldsymbol{n}}p_{\boldsymbol{n}}|\boldsymbol{n}|_1$ denotes the average total particle number which is finite. The corresponding maximal coherence is given by

$$C_r^{\max}=C_{r,d=1}^{\max}+\sum_{n=0}^{\infty}\frac{\bar{n}_t^n}{(\bar{n}_t+1)^{n+1}}\log_2\mathbb{C}_{n+d-1}^{d-1}, \tag{233}$$

with

$$C_{r,d=1}^{\max}=(\bar{n}+1)\log_2(\bar{n}+1)-\bar{n}\log_2\bar{n}, \tag{234}$$

being the maximal coherence for the single-mode case ($\bar{n}_t=\bar{n}$).

3.7.2. Analytic formulas

A state is said to be Gaussian if its characteristic function $\chi(\rho,\lambda)=\mathrm{tr}[\rho D(\lambda)]$ is Gaussian, where $D(\lambda)$ is the displacement operator. A Gaussian state is fully describable using the covariance matrix γ (with entries γ_{kl}) and displacement vector $\vec{\upsilon}=(\upsilon_1,\upsilon_2)^T$, $\rho=\rho(\gamma,\vec{\upsilon})$. The incoherent thermal state $\rho_{\rm th}(\bar{n})$ corresponds to $\gamma=(2\bar{n}+1)\mathbb{1}$ and $\vec{\upsilon}=(0,0)^T$, where the superscript T denotes transpose.

For the case of d-mode Gaussian states $\rho(\gamma,\vec{\upsilon})$, by denoting $x_i=[\det\gamma^{(i)}]^{1/2}$ square of the determinant of the covariance matrix $\gamma^{(i)}$ for the ith mode, Xu (2016) obtained analytical formula for the relative entropy of coherence, which is given by

$$\begin{aligned}C_r(\rho)=&\sum_{i=1}^{d}\left(\frac{x_i-1}{2}\log_2\frac{x_i-1}{2}-\frac{x_i+1}{2}\log_2\frac{x_i+1}{2}\right)\\&+\sum_{i=1}^{d}[(\bar{n}_i+1)\log_2(\bar{n}_i+1)-\bar{n}_i\log_2\bar{n}_i],\end{aligned} \tag{235}$$

where $\bar{n}_i$ can be written in terms of the covariance matrix $\gamma^{(i)}$ and displacement vector $\vec{\upsilon}$ as

$$\bar{n}_i=\frac{1}{4}\{\gamma_{11}^{(i)}+\gamma_{22}^{(i)}+[\upsilon_1^{(i)}]^2+[\upsilon_2^{(i)}]^2-2\}, \tag{236}$$

from which one can also see that the maximally coherent state should be pure, i.e., $x_i=1,\forall\, i\in\{1,\ldots,d\}$.

3.7.3. Coherence of coherent states

For the set $\{|\alpha\rangle\}$ of coherent states which spans an infinite-dimensional Hilbert space, a direct application of the resource theory of quantum coherence formulated by Baumgratz et al. (2014) is not applicable. This is because states of the considered set are not only overcomplete but also may be not linearly independent. To circumvent this perplexity, Tan et al. (2017) developed an orthogonalization procedure which allows for defining coherence measure for arbitrary superposition of coherent states, whether they are orthogonal or not.

For a given density operator ρ_A, let $\rho_{AB}^{(0)}=\rho_A\otimes|0^B\rangle\langle 0^B|$ and $\rho_{AB}^{(i)}=U_{\alpha^{(i)}}\rho_{AB}^{(i-1)}U_{\alpha^{(i)}}^\dagger$ $(i=1,\ldots,N)$. Then if one denotes $|\alpha^{(i)}\rangle$ for the coherent state admitting $\mathrm{tr}(|\alpha^{(i)}\rangle\langle\alpha^{(i)}|\otimes|0^B\rangle\langle 0^B|\rho_{AB}^{(i-1)})=\max_{\{|\alpha\rangle\}}\mathrm{tr}(|\alpha\rangle\langle\alpha|\otimes|0^B\rangle\langle 0^B|\rho_{AB}^{(i-1)})$, the Nth Gram–Schmidt unitary $U_{\rm GS}^{(N)}=U_{\alpha^{(N)}}\ldots U_{\alpha^{(1)}}$, where the CNOT type unitary is given by

$$\begin{aligned}U_{\alpha^{(i)}}=&\mathbb{I}\otimes\mathbb{I}+|\alpha^{(i)}\rangle\langle\alpha^{(i)}|\otimes(|\beta^{(i)}\rangle\langle 0^B|+|0^B\rangle\langle\beta^{(i)}|\\&-|0^B\rangle\langle 0^B|-|\beta^{(i)}\rangle\langle\beta^{(i)}|).\end{aligned} \tag{237}$$

Using the Gram–Schmidt unitary, Tan et al. (2017) defined the N coherence for a general state ρ_A as follows

$$C_\alpha(\rho_A,N)=\inf_{\substack{\rho_{AE}\in\mathcal{E}\\ U_{\rm GS}\in\mathcal{S}^{(N)}}}C(\Phi_{\rm GS}^{(N)}[\rho_{AE}]), \tag{238}$$

where C denotes any faithful coherence measure, $\mathcal{S}^{(N)}$ denotes the full set of Nth Gram–Schmidt unitaries, $\mathcal{E} = \{\rho_{AE} | \mathrm{tr}_E \rho_{AE} = \rho_A\}$ is the set of extensions of ρ_A, and

$$\Phi_{\mathrm{GS}}^{(N)}[\rho_A] = \frac{\Pi_{\mathrm{GS}}^{(N)}[U_{\mathrm{GS}}^{(N)}(\rho_A \otimes |0^B\rangle\langle 0^B|)U_{\mathrm{GS}}^{(N)\dagger}]\Pi_{\mathrm{GS}}^{(N)}}{\mathrm{tr}\{\Pi_{\mathrm{GS}}^{(N)}[U_{\mathrm{GS}}^{(N)}(\rho_A \otimes |0^B\rangle\langle 0^B|)U_{\mathrm{GS}}^{(N)\dagger}]\Pi_{\mathrm{GS}}^{(N)}\}}, \tag{239}$$

with the projector

$$\Pi_{\mathrm{GS}}^{(N)} = \sum_{i=1}^{N} |\alpha^{(i)}\rangle\langle\alpha^{(i)}| \otimes |\beta^{(i)}\rangle\langle\beta^{(i)}|. \tag{240}$$

Then, an ε-smoothed version of N coherence can be written as

$$C_\alpha^\varepsilon(\rho_A, N,) = \inf_{\rho_A' \in \mathcal{B}_\varepsilon(\rho_A)} C_\alpha(\rho_A', N), \tag{241}$$

where $\mathcal{B}_\varepsilon(\rho_A) = \{\rho_A' | \frac{1}{2}\|\rho_A' - \rho_A\|_1 \le \varepsilon\}$. Finally, the α coherence is defined as the smoothed N coherence in the asymptotic limit, that is,

$$C_\alpha(\rho_A) = \lim_{\varepsilon \to 0} \lim_{N \to \infty} C_\alpha^\varepsilon(\rho_A, N,). \tag{242}$$

Tan et al. (2017) proved that $C_\alpha(\rho_A) = 0$ if and only if ρ_A is a classical state, and it is also a nonclassicality measure.

3.8. Generalized coherence measures

When the constraints imposed by the axiomatic-like postulates of Baumgratz et al. (2014) are somewhat released, one may introduce other measures of quantum coherence that are physically relevant. These measures may also have potential applications under specific contexts.

3.8.1. Basis-independent coherence measure

For a state ρ in the d-dimensional Hilbert space, Yao et al. formulated the following basis-independent measure of quantum coherence

$$C_{BI}(\rho) = \sqrt{\frac{d}{d-1}} \|\rho - \rho_{\mathrm{mm}}\|_2, \tag{243}$$

which is proportional to the HS distance between ρ and the maximally mixed state $\rho_{\mathrm{mm}} = \mathbb{1}_d/d$, and the parameter before $\|\cdot\|_2$ is introduced for normalizing $C_{\mathrm{free}}(\rho)$.

The $C_{BI}(\rho)$ can be calculated analytically as

$$C_{BI}(\rho) = \sqrt{\frac{dP-1}{d-1}} = \sqrt{\frac{d\mathcal{I}_{BZ}}{d-1}}, \tag{244}$$

where $P = \mathrm{tr}\rho^2$ is the purity of ρ, and $P - 1/d$ equals the Brukner–Zeilinger invariant information

$$\mathcal{I}_{BZ}(\rho) = \sum_{i=1}^{m} \sum_{j=1}^{d} \left[\mathrm{tr}(\Pi_{ij}\rho) - \frac{1}{d}\right]^2, \tag{245}$$

thereby endows $C_{BI}(\rho)$ a clear physical meaning. Here, $\{\Pi_{ij}\}$ denote eigenvectors of the mutually complementary observables, for example, for the case of $d = 2$, they are those of the Pauli operators σ_x, σ_y, and σ_z.

The coherence measure $C_{BI}(\rho)$ is unitary invariant (a trait that distinguishes it from other coherence measures), takes the maximum 1 for all the pure states, and is nonincreasing under the action of any unital channel Λ_u, i.e., $C_{BI}(\rho) \ge C_{BI}(\Lambda_u[\rho])$. Moreover, it also provides loose lower bounds for the l_1 norm of coherence and trace norm of coherence, i.e.,

$$\begin{aligned} &C_{l_1}(\rho) \le \sqrt{d(d-1)} C_{BI}(\rho), \\ &C_{tr}(\rho) \le \sqrt{d-1} C_{BI}(\rho). \end{aligned} \tag{246}$$

As the measures of quantum coherence are basis dependent, it is of particular interest to consider the maximum amount of coherence attainable by varying the reference basis, and define

$$C^{\max}(\rho) = \max_U C(U\rho U^\dagger), \tag{247}$$

with C being any valid coherence measure and $\{U\}$ the set of unitary operations.

Yu et al. (2016a) investigated problems of such kind. For a d-dimensional state ρ, they proved

$$C_r^{\max}(\rho) = \log_2 d - S(\rho), \quad C_{l_2}^{\max}(\rho) = \mathrm{tr}\rho^2 - \frac{1}{d}, \tag{248}$$

where $C_r(\rho)$ and $C_{l_2}(\rho)$ denote, respectively, the relative entropy and the HS norm measure of coherence. In particular, $C_r^{\max}(\rho)$ equals the relative entropy between ρ and the corresponding maximally mixed state $\rho_{\rm mm}$, and $C_2^{\max}(\rho)$ equals the squared HS norm between ρ and $\rho_{\rm mm}$. Both $C_r^{\max}(\rho)$ and $C_2^{\max}(\rho)$ take the maximum value for pure states and zero for incoherent states. They also possess preferable features of coherence measures such as: (i) invariant under unitary operations; (ii) convexity under mixing of states; (iii) monotonicity under the unitary operations $\{U_i|\sum p_i U_i^\dagger U_i = \mathbb{1}, U_i^\dagger U_i = \mathbb{1}\}$ and SIO.

For some other widely used coherence measures including the RoC, the coherence weight, and the modified skew information measure of coherence, $C^{\max}(\rho)$ defined in Eq. (247) can also be obtained analytically, which reads (Hu et al., 2017)

$$
\begin{aligned}
&C_R^{\max}(\rho) = d\lambda_{\max} - 1, \;\; C_w^{\max}(\rho) = 1 - d\lambda_{\min}, \\
&C_{sk}^{\max}(\rho) = 1 - \frac{1}{d}\left(\sum_i \sqrt{\lambda_i}\right)^2.
\end{aligned}
\tag{249}
$$

where $\{\lambda_i\}$ denote the eigenvalues of ρ, with $\lambda_{\max} = \max\{\lambda_i\}$ and $\lambda_{\min} = \min\{\lambda_i\}$.

Streltsov et al. (2018) also considered the maximal coherence achievable by performing unitary operations on a state ρ, with however a different set of free operations from that of Baumgratz et al. (2014) was used when defining $C(\rho)$. To be explicit, they used the set of MIO $\Lambda_{\rm MIO}[\delta] \in \mathcal{I}$ $(\forall \delta \in \mathcal{I})$ instead of the traditional IO as the free operations, which was first suggested by Aberg (2006). Clearly, the set of IO is a subset of MIO, and thus any coherence monotone with respect to MIO is also a coherence monotone with respect to IO, but the inverse may not always be true, e.g., the l_1 norm of coherence and the coherence of formation is an IO monotone but not a MIO monotone (Hu, 2016).

Based on the above setting, Streltsov et al. (2018) showed that the following state

$$
\rho_{\max} = \sum_{n=1}^{d} p_n |n_+\rangle\langle n_+|,
\tag{250}
$$

is a MCMS with respect to any MIO monotone, where $\{p_n\}$ is the probability distribution, and $\{|n_+\rangle\}$ denotes a MUB with respect to the incoherent basis $\{|i\rangle\}$, i.e., $|\langle i|n_+\rangle| = 1/d, \forall i, n_+$ (Wootters, 1986; Wootters and Fields, 1989). Eq. (250) can be proved straightforwardly by noting that $\Lambda_{\rm MIO}[\rho_{\max}] = U\rho_{\max}U^\dagger$ if one chooses Kraus operators of $\Lambda_{\rm MIO}$ as $K_n = U|n_+\rangle\langle n_+|$. This is because $C(U\rho_{\max}U^\dagger) = C(\Lambda_{\rm MIO}[\rho_{\max}]) \leq C(\rho_{\max})$, where the inequality is due to the monotonicity of a coherence measure under MIO. By the way, one can also show that $\Lambda_{\rm MIO}$ of this type yields $\Lambda_{\rm MIO}[\delta] = \mathbb{1}/d$ and $\Lambda_{\rm MIO}[\rho] \in \rho_{\max}$.

Furthermore, when the coherence is measured by the shortest distance between ρ and the set $\mathcal{I}$ of incoherent states [see Eq. (143), with only the IO being replaced by MIO], we have

$$
C^{\max}(\rho) = C(\tilde{U}\rho\tilde{U}^\dagger) \leq \mathcal{D}(\tilde{U}\rho\tilde{U}^\dagger, \mathbb{1}/d) = \mathcal{D}(\rho, \mathbb{1}/d),
\tag{251}
$$

where $\tilde{U}$ is the optimal unitary for obtaining $C^{\max}(\rho)$, and the inequality comes from the fact that $\mathbb{1}/d$ is not necessary the closest incoherent state to $\tilde{U}\rho\tilde{U}^\dagger$, while the last equality is due to the unitary invariance of $\mathcal{D}$. Moreover, by denoting Δ_+ the full dephasing of ρ in the maximally coherent basis $\{n_+\rangle\}$ [see the definition in Eq. (142)], one can obtain $\Delta_+[\rho_{\max}] = \rho_{\max}$ and $\Delta_+[\delta] = \mathbb{1}/d$. Thus by denoting $\tilde{\delta}$ the closest incoherent state to $\rho_{\max}$, we have

$$
\begin{aligned}
C^{\max}(\rho) &\geq C(\rho_{\max}) = \mathcal{D}(\rho_{\max}, \tilde{\delta}) \\
&\geq \mathcal{D}(\Delta_+[\rho_{\max}], \Delta_+[\tilde{\delta}]) \\
&= \mathcal{D}(\rho_{\max}, \mathbb{1}/d) = \mathcal{D}(\rho, \mathbb{1}/d),
\end{aligned}
\tag{252}
$$

then by combining the above two equations, one can obtain

$$
C^{\max}(\rho) = C(\rho_{\max}) = \mathcal{D}(\rho, \mathbb{1}/d),
\tag{253}
$$

for any contractive distance measure $\mathcal{D}$ of two states.

When ρ is incoherent, i.e., $\rho = \delta$, Eq. (247) corresponds to the maximum amount of coherence (quantified by any faithful measures) generated from a given incoherent state δ. By focusing on the two-qubit states only, and arranging δ's diagonal elements as $\delta_1 \leq \delta_2 \leq \delta_3 \leq \delta_4$ and taking the relative entropy as a measure of coherence, Yao et al. (2015) obtained solutions of Eq. (247) for specified types of U. They are

$$
\begin{aligned}
&C_{r,1}^{\max} = 1 + H(\delta_1 + \delta_3) - \sum_i \delta_i \log_2 \delta_i, \\
&C_{r,2}^{\max} = 2 - \sum_i \delta_i \log_2 \delta_i,
\end{aligned}
\tag{254}
$$

where $C_{r,1}^{\max}$ ($C_{r,2}^{\max}$) is the optimal coherence created under local unitaries $U_A \otimes \mathbb{1}$ ($U_A \otimes U_B$), with the corresponding

$$
U_A = U_B = |+\rangle\langle 1| + |-\rangle\langle 0|,
\tag{255}
$$

where $|\pm\rangle = (|0\rangle \pm |1\rangle)/\sqrt{2}$, and $H(\cdot)$ is the binary Shannon entropy. For the kernel U_d of nonlocal U in the Cartan decomposed form, Yao et al. (2015) found that they cannot outperform the local unitaries on creating quantum coherence, but it remains open if this is also true for more general nonlocal U.

3.8.2. Genuine quantum coherence

The core for the resource theory of quantum coherence is incoherent states and incoherent operations. de Vicente and Streltsov (2017) introduced a slightly different types of incoherent operations which he called genuinely incoherent operations (GIO). These operations preserve all incoherent states, i.e.,

$$\Lambda_{\mathrm{gi}}(\delta) = \delta, \tag{256}$$

for all $\delta \in \mathcal{I}$. This definition of GIO ensures their independency on the explicit forms of the Kraus decompositions.

It has been proved that the Kraus operators of Λ_{gi} are diagonal in the prefixed reference basis, and

$$\Lambda_{\mathrm{gi}}(\cdot) = \sum_k p_k U_k(\cdot) U_k^\dagger, \tag{257}$$

for the single-qubit case, with $\{p_k\}$ being the probability distribution and $U_k = \sum_l e^{i\phi_{lk}} |k\rangle\langle k|$ the unitary. Moreover, the GIO includes the nondegenerate thermal operations Λ_{th} as a special case, while itself belongs to the set of translationally invariant operations Λ_{ti} (de Vicente and Streltsov, 2017). Given a system state ρ^S with the Hamiltonian $\hat{H}_S$, and the thermal state $\rho_{\mathrm{th}}^E = e^{-\beta \hat{H}_E}/\mathrm{tr} e^{-\beta \hat{H}_E}$ with $\hat{H}_E$ the environmental Hamiltonian and $\beta = 1/T$ the inverse temperature, Λ_{th} and Λ_{ti} are defined by

$$\begin{aligned} &\Lambda_{\mathrm{th}}(\rho^S) = \mathrm{tr}_E(U\rho^S \otimes \rho_{\mathrm{th}}^E U^\dagger), \\ &\Lambda_{\mathrm{ti}}(e^{-i\hat{H}_S t} \rho e^{i\hat{H}_S t}) = e^{-i\hat{H}_S t} \Lambda_{\mathrm{ti}}(\rho) e^{i\hat{H}_S t}, \end{aligned} \tag{258}$$

where U is the unitary which preserves the total energy of the considered system plus its environment.

Under the set of GIO, de Vicente and Streltsov (2017) proposed the prerequisites for a function to be a genuine coherence measure. They are analogous to those labeled as (C1), (C2a), (C2b), and (C3), with only the IO being replaced by the GIO. de Vicente and Streltsov (2017) called a measure respecting the first three prerequisites a genuine coherence monotone. As the GIO is a strict subset of the general IO, the l_1 norm, relative entropy, and intrinsic randomness measures of coherence, as well as the RoC are all genuine coherence monotones.

Apart from the above measures, the WY skew information $\mathrm{I}(\rho, K)$ obeys (C1), (C2a), and (C3). The distance-based measure of genuine coherence

$$G_p(\rho) = \min_{\delta \in \mathcal{I}} \|\rho - \delta\|_p, \tag{259}$$

also obeys these three conditions, and for the special case of $p = 2$, $G_2(\rho) = \|\rho - \Delta(\rho)\|_2$, where the closest incoherent state is $\Delta(\rho) = \sum_i \langle i|\rho|i\rangle |i\rangle\langle i|$. For other cases, $\Delta(\rho)$ is not the closest state for obtaining $G_p(\rho)$, but de Vicente and Streltsov (2017) showed that

$$\tilde{G}_p(\rho) = \|\rho - \Delta(\rho)\|_p, \tag{260}$$

is also a valid genuine coherence measure, as it obeys the conditions (C1), (C2a), and (C3).

3.8.3. Quantification of superposition

As a generalization of the resource theories of coherence, Theurer et al. (2017) introduced a similar framework for quantifying superposition. In their framework, the set $\mathcal{F}$ of free states is comprised of the states that can be represented as statistical mixtures of linear independent (not necessarily orthogonal) basis states $\{|c_i\rangle\}_{i=1}^d$. To be explicit, these superposition-free states are given by

$$\varsigma = \sum_{i=1}^d \varsigma_i |c_i\rangle\langle c_i|, \tag{261}$$

where $\varsigma_i \geq 0$ and $\sum_i \varsigma_i = 1$. Those states that are not free are called superposition states. Similarly, the quantum operation $\Phi(\rho) = \sum_i K_i \rho K_i^\dagger$ is said to be superposition-free if the Kraus operator gives the map $K_i \varsigma K_i^\dagger / \mathrm{tr}(K_i \varsigma K_i^\dagger) \in \mathcal{F}$ ($\forall K_i$), that is, every K_i (hence Φ) maps the superposition-free state to another superposition state. Theurer et al. (2017) showed that such a K_i is of the following general form:

$$K_i = \sum_k c_i(k) |c_{f_i(k)}\rangle\langle c_k^\perp|, \tag{262}$$

where $|c_k^\perp\rangle$ are called reciprocal states which satisfy $\langle c_k^\perp|c_l\rangle = \delta_{kl}$, $c_i(k)$ are coefficients, and $f_i(k)$ are index functions.

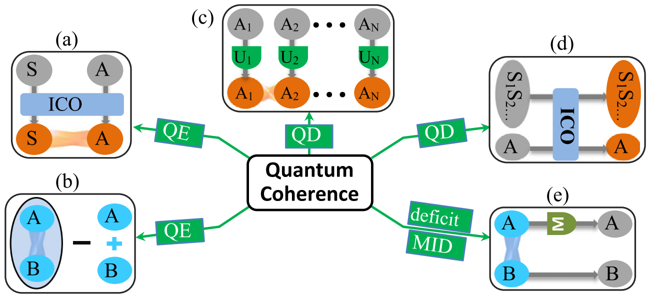

Fig. 4. Schematic picture for the connections between quantum coherence and quantum correlations. (a) Quantum entanglement (QE) based measure of quantum coherence [352], (b) correlated coherence and QE [354], (c) basis-free coherence measure and relative entropy of QD [412], (d) quantum coherence consumption and discord generation [242], (e) difference of quantum coherence for the premeasurement and postmeasurement states and one-way quantum deficit [404], as well as steering-induced coherence and MID (in the basis spanned by local eigenvectors of ρ^B) [174].

Theurer et al. (2017) presented the defining conditions for a faithful superposition measure $M(\rho)$. These conditions are very similar to those for a faithful coherence measure (Baumgratz et al., 2014). The difference is that $\Lambda(\rho)$ and δ in (C1), (C2a), (C2b), and C(3) were replaced by $\Phi(\rho)$ and ς, respectively. Then, in a similar manner to the definition of $C_{\mathcal{D}}(\rho)$ in Eq. (143), one can define

$$M_{\mathcal{D}}(\rho) = \min_{\varsigma\in\mathcal{F}} D(\rho, \varsigma). \tag{263}$$

For explicit distance measures, Theurer et al. (2017) proved the superposition measures including the relative entropy of superposition, the l_1 norm of superposition, and the robustness of superposition. They are similar to the coherence measures $C_r(\rho)$, $C_{l_1}(\rho)$, and $C_R(\rho)$ defined respectively, in Eqs. (144), (147), and (194). Apart from these, Theurer et al. (2017) also proved the rank-measure of superposition

$$M_{\mathrm{rank}}(\rho) = \min_{\{p_i,\psi_i\}} \sum_i p_i \log[r_S(\psi_i)], \tag{264}$$

where the superposition rank $r_S(\psi_i)$ is the number of nonzero $\alpha_n^{(i)}$ for $|\psi_i\rangle = \sum_n \alpha_n^{(i)}|c_n\rangle$, and the minimization is taken over all pure state decompositions of ρ showed in Eq. (13).

Before ending this section, we remark here that while various coherence measures have been introduced, Zhang et al. (2018) proposed a proposal for estimating their values with limited experimental data available. Their approach is based on the optimization of a Lagrangian function and the limited expectation value of certain Hermitian operators, and can be applied to any coherence measure $C(\rho)$ that is continuous and convex.

4. Interpretation of quantum coherence

Coherence is not only a basic feature signifying quantumness in an integral system, but also a common prerequisite for different forms of quantum correlations when composite systems are considered. Apart from its characterization and quantification, it has also been shown to be intimately related to many other quantities manifesting quantumness of states, and fundamental problems of quantum mechanics such as complementarity and uncertainty relations. All these have triggered the community's interest in investigating it from different perspectives, which endows quantum coherence clear operational interpretations and physical meanings.

4.1. Coherence and quantum correlations

In the seminal work of Baumgratz et al. (2014) and the subsequent stream of works, the quantum coherence measures are defined for single systems. Contrary to it, the traditional manifestation of quantumness for a system, e.g., quantum correlations, are defined in a scenario which involves at least two parties. In fact, both quantum coherence and quantum correlations arise from the superposition principle of quantum mechanics, hence it is essential to study the interrelation between them. The main progresses up to now are summarized in Fig. 4, and we review them in detail in the following.

4.1.1. Coherence and entanglement

Streltsov et al. (2015) made a first step toward the above problem. By considering the setting where a coherent state ρ^S is coupled to an incoherent ancilla initially in the vacuum state $|0^A\rangle$, they showed that the generated entanglement between S and A is upper bounded by the coherence of S. The bound can be saturated for certain contractive distance measures, hence yields a natural family of entanglement-based coherence measures, see Section 2.2 for more detail.

Qi et al. (2017) considered a very similar problem to that of Streltsov et al. (2015). They used the coherence concurrence and entanglement concurrence, and found that the generated entanglement concurrence from the initial state $\rho^S \otimes |0^A\rangle\langle 0^A|$ is upper bounded by

$$C_E(\Lambda^{SA}[\rho^S \otimes |0^A\rangle\langle 0^A|]) \leq C_{con}(\rho^S), \tag{265}$$

and when ρ^S is a two-qubit state while the ancilla A is also a qubit, the above equality is saturated. Moreover, by applying the generalized CNOT gate of Eq. (170), they also found a lower bound of the created entanglement

$$C_E(\Lambda^{SA}[\rho^S \otimes |0^A\rangle\langle 0^A|]) \geq \sqrt{\frac{2}{d(d-1)}} C_{\text{con}}(\rho^S). \tag{266}$$

Apart from the link to entanglement, Tan et al. (2016) introduced the concept of correlated coherence, and argued that it can be connected to QD and entanglement. Their key idea is by distinguishing the coherence in ρ^{AB} as local and nonlocal, i.e., by dividing the total coherence into two different portions: those stored locally in the subsystems, and those stored only in the correlated states. Based on this starting point, they defined the correlated coherence as

$$C_{cc}(\rho^{AB}) = C(\rho^{AB}) - C(\rho^A) - C(\rho^B), \tag{267}$$

which is a nonnegative quantity.

By choosing tensor products of the eigenvectors $\{|i_A\rangle\}$ (for ρ^A) and $\{|j_B\rangle\}$ (for ρ^B) as reference basis (for degenerate case, they will be chosen to be those minimize C_{cc}), $C_{cc}(\rho^{AB}) = 0$ if and only if $\rho^{AB} \in \mathcal{CC}$. Similarly, $C_{cc}(\rho^{AB}) = C_{cc}(\Delta^A[\rho^{AB}])$ if and only if $\rho^{AB} \in \mathcal{CQ}$. Based on these observations, Tan et al. (2016) defined

$$E_{cc}(\rho^{AB}) := \min C_{cc}(\rho^{AA'BB'}), \tag{268}$$

and showed that it possesses the preferable properties for an entanglement monotone. Here, the minimization is taken over the full set of unitarily symmetric extensions of ρ^{AB} satisfying

$$U_{AA'} \otimes U_{BB'}(U_{\text{SWAP}}\rho^{AA'BB'}U_{\text{SWAP}}^\dagger)U_{AA'}^\dagger \otimes U_{BB'}^\dagger = \rho^{AA'BB'}, \tag{269}$$

where $\rho^{AB} = \text{tr}_{A'B'}\rho^{AA'BB'}$ for all the local unitaries $U_{AA'}$ and $U_{BB'}$, and U_{SWAP} is the swap operator.

Mondal et al. (2017) examined the steered coherence from another perspective. In their framework, Alice and Bob hold respectively, qubit A and B of ρ^{AB}, and agree on the observables $\{\sigma_1, \sigma_2, \sigma_3\}$ in advance. Alice then measures σ_i on her qubit and informs Bob of her choice σ_i and outcome $a \in \{0, 1\}$. Bob computes the coherence of his conditional states $\{p(a|\sigma_i), \rho_{B|\sigma_i^a}\}$ in the eigenbasis of either σ_j or σ_k ($j, k \neq i$) randomly, which can be written as $\sum_a p(a|\sigma_i)C^{\sigma_j}(\rho_{B|\sigma_i^a})$. Here, $p(a|\sigma_i)$ is the probability for Alice's outcome a when she measures σ_i, and $\rho_{B|\sigma_i^a}$ is the corresponding postmeasurement state of B. By averaging over Alice's possible measurements and Bob's allowable eigenbasis sets, one can obtain

$$\begin{aligned} C_{l_1}^{na}(\rho^{AB}) &= \frac{1}{2}\sum_{\substack{i,j,a\\ i\neq j}} p(a|\Pi_i)C_{l_1}^{\sigma_j}(\rho_{B|\sigma_i^a}),\\ C_r^{na}(\rho^{AB}) &= \frac{1}{2}\sum_{\substack{i,j,a\\ i\neq j}} p(a|\Pi_i)C_{re}^{\sigma_j}(\rho_{B|\sigma_i^a}). \end{aligned} \tag{270}$$

As for any single-partite state ρ, we have

$$\begin{aligned} &\sum_{j=1}^{3} C_{l_1}^{\sigma_j}(\rho) \leq \sqrt{6},\\ &\sum_{j=1}^{3} C_r^{\sigma_j}(\rho) \leq C_2^m = 3H(1/2+\sqrt{3}/6), \end{aligned} \tag{271}$$

it is said that a nonlocal advantage of quantum coherence is achieved on B when $C_{l_1}^{na}(\rho^{AB}) > \sqrt{6}$ or $C_r^{na}(\rho^{AB}) > C_2^m$. Mondal et al. (2017) showed that any two-qubit state that can achieve a nonlocal advantage of quantum coherence is quantum entangled. Moreover, the interplay between nonlocal advantage of quantum coherence and Bell nonlocality was also established for two-qubit states (Hu et al., 2018).

The above framework was extended to $(d \times d)$-dimensional state ρ^{AB}, in which the Pauli observables are replaced by the set of mutually unbiased observables $\{A_i\}$ (Hu and Fan, 2018). Now, the average coherence for Bob's conditional states is

$$C^{na}(\rho_{AB}) = \frac{1}{d}\sum_{\substack{i,j,a\\ i\neq j}} p(a|A_i)C^{A_j}(\rho_{B|A_i^a}), \tag{272}$$

and $C^{na}(\rho_{AB}) > C^m$ captures the existence of nonlocal advantage of quantum coherence in state ρ_{AB}. When one adopts the l_1 norm of coherence and relative entropy of coherence, the state-independent bound C^m is given by $(d-1)\sqrt{d(d+1)}$ and $(d+1)\log_2 d-(d-1)^2\log_2(d-1)/d(d-2)$, respectively (Hu and Fan, 2018).

Similarly, one can also formulate other framework for capturing the nonlocal advantage of quantum coherence in a state, e.g., after Alice executing one round of measurements and announced her choice A_i and outcomes a, Bob can measure the coherence of his conditional states only in the preagreed basis spanned by the eigenvectors of A_{α_i}, with $\{\alpha_i\}$ being any one of the possible permutations of the elements of $\{i\}$. This can give some new insights on the interrelation between coherence and quantum correlations (Hu and Fan, 2018).

4.1.2. Coherence and quantum discord

Since the coherence measures reviewed in Section 3 are basis dependent, they can be changed by unitary operations. Based on this consideration, Yao et al. (2015) introduced a basis-free coherence measure of the following form

$$C^{\text{free}}(\rho)=\min_{\vec{U}} C(\boldsymbol{U}\rho\boldsymbol{U}), \tag{273}$$

where $\boldsymbol{U}=U_1\otimes U_2\otimes\cdots\otimes U_N$ for a N-partite state ρ. It is in fact the minimum coherence created by local unitary operations.

By putting the measures of coherence and QD on an equal footing, that is, to quantify the both via relative entropy, Yao et al. (2015) found that $C^{\text{free}}(\rho)$ defined above equals the QD $D_r(\rho)=\min_{\chi\in\mathcal{C}}S(\rho\parallel\chi)$ ($\mathcal{C}$ is the set of classical states) introduced by Modi et al. (2010), i.e.,

$$C^{\text{free}}(\rho)=D_r(\rho), \tag{274}$$

thereby establishes a direct connection between coherence of a N-partite state in the product basis $\{|\boldsymbol{i}\rangle:=\otimes_{k=1}^N|i_k\rangle\}$ (with $\{|i_k\rangle\}_{i=1}^{d_k}\}$ and $d_k=\dim\mathcal{H}_k$) and QD of this state with the same multipartite divisions.

For bipartite states ρ and local von Neumann measurements $\{\Pi_k^A\}$, Xi and Li (2015) found that the difference between relative entropy of coherence for ρ and the postmeasurement state $\rho'=\sum_k(\Pi_k^A\otimes\mathbb{1}_B)\rho(\Pi_k^A\otimes\mathbb{1}_B)$ equals the one-way quantum deficit (Oppenheim et al., 2002; Horodecki et al., 2005b), i.e.,

$$C_r(\rho)-C_r(\rho')=\Delta^{\rightarrow}(\rho). \tag{275}$$

For a system with N-partite division $S=\{S_1,S_2,\ldots,S_N\}$ and product reference basis $\{|\boldsymbol{i}\rangle\}$, one can define the sets of S_k-incoherent states $\mathcal{I}_{S_k}$ with elements $\delta^{S_k}=\sum_i p_i|i_k\rangle\langle i_k|\otimes\rho_{\tilde{S}|k}$ ($\tilde{S}=S-S_k$), incoherent operations $\{\Lambda^{S_k}\}$ which map $\mathcal{I}_{S_k}$ to itself, and coherence measure $C_{\mathcal{D}}^{S_k}(\rho)=\min_{\delta\in\mathcal{I}_{S_k}}\mathcal{D}(\rho,\delta)$, all with respect to the local basis $\{|i_k\rangle\}$. When $\mathcal{D}$ is the relative entropy, we have

$$C_r^{S_k}(\rho)=S(\rho\parallel\Delta^{S_k}[\rho]), \tag{276}$$

where $\Delta^{S_k}[\rho]$ is the full dephasing of $\rho=\rho^{S_1S_2\cdots S_N}$ in the basis $\{|i_k\rangle\}$ of party S_k. The GQD $D(\rho):=\min_{\chi\in\mathcal{C}}\mathcal{D}(\rho,\chi)$, and for $\mathcal{D}$ to be the relative entropy, $D(\rho)=\min_\Delta S(\rho\parallel\Delta[\rho])$, with $\Delta=\otimes_{k=1}^N\Delta^{S_k}$ (Modi et al., 2012). Moreover, the global discord

$$\bar{D}(\rho)=\min_{\Delta}[S(\rho\parallel\Delta[\rho])-\sum_k S(\rho^{S_k}\parallel\Delta^{S_k}[\rho^{S_k}])], \tag{277}$$

and the usual asymmetric discord

$$D_{\tilde{S}|S_k}(\rho)=\min_{\Delta^{S_k}}[S(\rho\parallel\Delta^{S_k}[\rho])-S(\rho^{S_k}\parallel\Delta^{S_k}[\rho^{S_k}])]. \tag{278}$$

By considering an analogous setting to that constructed by Streltsov et al. (2015), i.e., the system S and an incoherent ancilla is prepared initially in the product state $\rho^S\otimes|0^A\rangle\langle 0^A|$, Ma et al. (2016a) studied, from the perspective of coherence consumption and discord generation, the interplay between quantum coherence and QD. First, they found that

$$D(\Lambda^{SA}[\rho^S\otimes\rho^A])\le C_{\mathcal{D}}(\rho^S), \tag{279}$$

for any contractive measure of $\mathcal{D}$, i.e., the generated discord is upper bounded by the initial coherence in ρ^S. In particular, if $d_A\ge d_S$, the equality holds for $\mathcal{D}$ to be the relative entropy or the Bures distance. Second, for $\rho^{S_1S_2\cdots S_N}=\otimes_{k=1}^N\rho^{S_k}$, the sum of coherence consumed for all subsystems bounding the amount of global discord that can be generated by IO, i.e.,

$$\bar{D}(\Lambda[\rho^{S_1S_2\cdots S_N}])\le\sum_k\delta C_r(\rho^{S_k}), \tag{280}$$

where $\delta C_r=C_r^i-C_r^t$, with C_r^i (C_r^t) being coherence of the state prior to the measurement (after the measurement). Similarly, for the asymmetric discord,

$$D_{S_2|S_1}(\Lambda^{S_1}[\rho^{S_1S_2}])\le\delta C(\rho^{S_1}), \tag{281}$$

for $\rho^{S_1S_2}=\rho^{S_1}\otimes\rho^{S_2}$.

In a recent paper, Hu and Fan (2017) introduced the concept of relative quantum coherence (RQC), which is the coherence of one state in the reference basis spanned by the eigenvectors of another one. To be explicit, for ρ and σ in the same Hilbert space $\mathcal{H}$, and the eigenvectors of σ being given by $\{|\psi_i\rangle\}$, with the corresponding eigenvalues $\{\epsilon_i\}$, the RQC is given by

$$C(\rho,\sigma)=C^{\Xi}(\rho), \tag{282}$$

where $C^{\Xi}(\rho)$ denotes any *bona fide* measure of quantum coherence defined in the reference basis Ξ.

When the quantum coherence is measured by the l_1 norm, they showed that the QD $D_A(\rho_{AB})$ (Ollivier and Zurek, 2001) is bounded from above by the discrepancy between the RQC for the total system and that for the subsystem to be measured in the definition of QD, that is

$$D_A(\rho_{AB})\leqslant C_{\rm re}(\rho_{AB},\tilde{\rho}_{\mathbb{PQ}})-C_{\rm re}(\rho_A,\tilde{\rho}_{\mathbb{P}}), \tag{283}$$

where $\tilde{\rho}_{\mathbb{PQ}}$ denotes the optimal postmeasurement state for obtaining the QD, and $\tilde{\rho}_{\mathbb{P}}$ is the reduced state of $\tilde{\rho}_{\mathbb{PQ}}$. This upper bound can also be saturated when the state ρ_{AB} is quantum–classical correlated.

Similarly, for the symmetric QD $D_s(\rho_{AB})=I(\rho_{AB})-I(\tilde{\rho}_{\mathbb{PQ}})$ defined via two-sided optimal measurements $\{\tilde{\Pi}_k^A\otimes\tilde{\Pi}_l^B\}$ (Wu et al., 2009; Girolami et al., 2011), it was further showed that (Hu and Fan, 2017)

$$D_s(\rho_{AB})=C_{\rm re}(\rho_{AB},\tilde{\rho}_{\mathbb{PQ}})-C_{\rm re}(\rho_A,\tilde{\rho}_{\mathbb{P}})-C_{\rm re}(\rho_B,\tilde{\rho}_{\mathbb{Q}}), \tag{284}$$

which implies that $D_s(\rho_{AB})$ is nonzero if and only if there exists RQC not localized in the subsystems. This establishes a direct connection between the RQC discrepancy and the symmetric discord.

For a bipartite system AB described by the density operator ρ^{AB}, Hu et al. (2015) considered the maximum amount of coherence created at party B by the procedure of steering on A, and defined the steered coherence

$$C_{\rm str}(\rho^{AB}):=\inf_{e^B}[\max_{E^A}C_{l_1}(e^B,\rho_i^B)], \tag{285}$$

where $E^A=\{E_k^A\}$ represents the set of POVM operators, and $\rho_i^B={\rm tr}_A(E_i^A\otimes\mathbb{1}_B\rho^{AB})/p_i$, $p_i={\rm tr}(E_i^A\otimes\mathbb{1}_B\rho^{AB})$. The infimum over the eigenbasis $e^B=\{e_k^B\}$ of ρ^B is necessary only when it is degenerate. The motivation for the definition of $C_{\rm str}(\rho^{AB})$ is very similar to the concept of localizable entanglement which is indeed the maximum entanglement that can be localized, on average, between two parties of a multipartite system, by performing local measurements on the other parties (Popp et al., 2005; Verstraete and Popp, 2004).

The steered coherence is shown to have several preferable properties of $C_{\rm str}(\rho^{AB})$, e.g., it vanishes when ρ^{AB} is quantum–classical correlated, takes the maximum for all pure entangled states with full Schmidt rank d_B, and is locally unitary invariant. Moreover, $C_{\rm str}(\rho^{AB})$ may be increased by the local operations Λ_B on B prior to the steering on A. For two-qubit states, this is achievable if and only if Λ_B is neither unital nor semi-classical. All these properties are very similar to those of the various discordlike quantum correlation measures (Modi et al., 2012).

For two-qubit states ρ^{AB}, the maximum steered coherence is given by (Hu et al., 2015)

$$C_{\rm str}(\rho^{AB})=\inf_{\vec{n}_B\in\mathbb{R}^3,|\vec{n}_B|=1}\left\{\max_{\vec{m}\in\mathbb{R}^3,|\vec{m}|=1}\left|\frac{R^T\vec{m}\times\vec{n}_B}{1+\vec{a}\cdot\vec{m}}\right|\right\}, \tag{286}$$

where R is a 3×3 matrix with elements $R_{ij}={\rm tr}\rho^{AB}(\sigma_i\otimes\sigma_j)$, $\vec{m}$ is a vector related to the POVM $E^A=(\mathbb{1}+\vec{m}\cdot\vec{\sigma})/2$, $\vec{a}={\rm tr}(\rho^A\vec{\sigma})$ is the local Bloch vector of ρ^A (similarly for $\vec{b}$), and $\vec{n}_B=\vec{b}/|\vec{b}|$.

4.1.3. Coherence and measurement-induced disturbance

Based on quantum steering, Hu and Fan (2016b) introduced the steering-induced coherence, and explored its connection with a quantum correlation measure known as measurement-induced disturbance (Luo, 2008). For a bipartite system described by density operator ρ^{AB} and shared by two players Alice and Bob, if Alice performs a local quantum measurement $\Xi^A=\{|\xi_i^A\rangle\langle\xi_i^A|\}$ on her subsystem, she will obtain the outcome i with probability $p_i={\rm tr}(\Xi_i^A\otimes\mathbb{1}_B\rho)$, and Bob's subsystem is steered to $\rho_i^B={\rm tr}_A(\Xi_i^A\otimes\mathbb{1}_B\rho)/p_i$. After multi-rounds of measurements, Bob will have the ensemble $\{p_i,\rho_i^B\}$. See Fig. 4(e) for an illustration of the scheme.

From the above scheme, Hu and Fan (2016b) defined the steering-induced coherence as

$$\bar{C}(\rho^{AB})=\inf_{e^B}[\max_{\Xi^A}\sum_i p_iC(e^B,\rho_i^B)], \tag{287}$$

where Bob's reference basis is chosen to be eigenbasis $e^B=\{|e_k^B\rangle\}$ of ρ^B, and the infimum in Eq. (287) is necessary only when ρ^B is degenerate. $\bar{C}(\rho)$ characterizes Alice's ability to steer coherence on Bob's side, and has also been proven satisfying the necessary requirements of a faithful coherence measure.

Based on the observation that the symmetric measurement-induced disturbance equals coherence of ρ^{AB} in the tensor-product eigenbasis of ρ^A and ρ^B, Hu and Fan (2016b) further considered the asymmetric measurement-induced disturbance

$$Q_B(\rho^{AB})=\inf_{E^B}\mathcal{D}(\rho^{AB},E^B[\rho^{AB}]), \tag{288}$$

with $E^B=\{|e_k^B\rangle\langle e_k^B|\}$ the locally invariant projective measurements on B. Their results show that when being quantified by the same distance measure, $\bar{C}(\rho^{AB})$ is bound from above by $Q_B(\rho^{AB})$, i.e.,

$$\bar{C}(\rho^{AB}) \le Q_B(\rho^{AB}), \tag{289}$$

and the equality holds for the maximally correlated state

$$\rho_{\rm mc} = \sum_{i,j} \rho_{ij} |ii\rangle\langle jj|, \tag{290}$$

when the relative entropy quantifiers of them are adopted, for which they both equal to $S(\rho_{\rm mc}^B)-S(\rho_{\rm mc}^{AB})$. Moreover, for the two-qubit state and the l_1 norm quantifiers, the upper bound is also saturated.

Zhang et al. (2017b) considered a very similar coherence steering scheme to that of Hu and Fan (2016b). The difference is that they used the computation basis $\{|i\rangle\}_{i=1}^d$, and discussed the amount of coherence gain with classical correlation of the premeasurement state ρ^{AB}. They defined the measurement-induced average coherence $\bar{C}_r^P$ and measurement-induced average total coherence $\bar{C}_r^T$ as

$$\begin{aligned} \bar{C}_r^P(\rho^B) &= \sum_i p_i C_r(\{|i\rangle\}, \rho_i^B), \\ \bar{C}_r^T(\rho^B) &= \sum_i p_i C_r^{\max}(\rho_i^B), \end{aligned} \tag{291}$$

where $C_r^{\max}(\rho_i^B)$ is the maximal attainable relative entropy of coherence under generic basis, see Eq. (146). The corresponding coherence gain is given by

$$\begin{aligned} \Delta C_r^P &= \bar{C}_r^P(\rho^B) - C_r(\rho^B), \\ \Delta C_r^T &= \bar{C}_r^T(\rho^B) - C_r^T(\rho^B). \end{aligned} \tag{292}$$

Based on these definitions, they found that

$$\Delta C_r^P \le \Delta C_r^T, \tag{293}$$

which can be seen from $\Delta C_r^P - \Delta C_r^T = \sum_i p_i S((\rho_i^B)_{\rm diag}) - S(\rho_{\rm diag}^B) \ge 0$. Moreover, the two coherence gains are proved to be upper bounded by the classical correlation (with respect to subsystem A) present in the premeasurement state ρ^{AB}, i.e.,

$$\max\{\Delta C_r^P, \Delta C_r^T\} \le J_A(\rho^{AB}), \tag{294}$$

where $J_A(\rho^{AB}) := S(\rho^B) - \min_{\{E_k^A\}} \sum_k q_k S(\rho_{B|E_k^A})$, and $\{E_k^A\}$ represents local positive operator valued measurements for defining classical correlation and QD (Ollivier and Zurek, 2001). Here, we provide a slightly different proof of the above equation from Zhang et al. (2017b). First, $\Delta C_r^T \le J_A(\rho^{AB})$ can in fact be obtained directly from $\Delta C_r^T = S(\rho^B) - \sum_i p_i S(\rho_i^B)$ and the ensemble $\{p_i, S(\rho_i^B)\}$ obtained with the measurement operators Ξ^A may not be optimal for attaining $J_A(\rho^{AB})$. Second, $\Delta C_r^P \le J_A(\rho^{AB})$ is due to Eq. (293).

Moreover, when a maximization process is performed over all possible Ξ^A, just like that of Eq. (287), the statements in the above two equations still hold. The optimized coherence gain ΔC_r^P equals zero if and only if $\rho^{AB} = \sum_i A_{ii} \otimes |i^A\rangle\langle i^A|$ [cf. Eq. (46) for the meaning of A_{ij}] or a product state, while ΔC_r^T equals zero if and only if $\rho^{AB} = \rho^A \otimes \rho^B$.

4.1.4. Distribution of quantum coherence

The distribution of quantum coherence among the subsystems of a multipartite system is also an interesting research direction. In fact, for quantum correlation measures such as entanglement and QD, similar problems have been studied via various monogamy inequalities, see, e.g., the works of Bai et al. (2014); Osborne and Verstraete (2006); Streltsov et al. (2012). For different coherence measures, one can also derive monogamy inequalities that impose limits on their shareability among multipartite systems. For example, for the l_1 norm of coherence, it is direct to show that

$$C_{l_1}(\rho^{A_1A_2\cdots A_N}) \ge \sum_i C_{l_1}(\rho^{A_i}). \tag{295}$$

for any multipartite system described by the density operator $\rho^{A_1A_2\cdots A_N}$, with ρ^{A_i} $(i=1,2,\ldots,N)$ being the reduced density operators.

For bipartite state ρ^{AB} with the reduced states $\rho^A = {\rm tr}_B \rho^{AB}$ and $\rho^B = {\rm tr}_A \rho^{AB}$, Xi and Li (2015) proved that the relative entropy of coherence respects the monogamy relation

$$C_r(\rho^{AB}) \ge C_r(\rho^A) + C_r(\rho^B). \tag{296}$$

In fact, for the multipartite system described by the density operator $\rho^{A_1A_2\cdots A_N}$, an application of Eq. (296) immediately yields

$$C_r(\rho^{A_1A_2\cdots A_N}) \geq \sum_i C_r(\rho^{A_i}). \tag{297}$$

But a similar monogamy relation does not hold for the general bipartite division of tripartite states. Even for the pure state $\rho^{ABC} = |\psi\rangle^{ABC}\langle\psi|$ with $\rho^{AB} = \mathrm{tr}_C\rho^{ABC}$ and $\rho^{AC} = \mathrm{tr}_B\rho^{ABC}$, the relation

$$C_r(\rho^{ABC}) \geq C_r(\rho^{AB}) + C_r(\rho^{AC}), \tag{298}$$

holds with a very strong constraint, that is, there should exist some real-valued parameters $\lambda \in [0, 1]$ such that (Liu et al., 2016b)

$$\lambda S(\rho^{AB}_{\mathrm{diag}}) \leq S(\rho^{AB}),\ \ (1-\lambda)S(\rho^{AC}_{\mathrm{diag}}) \leq S(\rho^{AC}). \tag{299}$$

Radhakrishnan et al. (2016) also explored the distribution of quantum coherence among constituents of a N-partite system. By introducing a quantum version of the Jensen–Shannon divergence (QJSD):

$$\mathcal{J}(\rho,\sigma) = \frac{1}{2}[S(\rho\|(\rho+\sigma)/2) + S(\sigma\|(\rho+\sigma)/2)], \tag{300}$$

or equivalently,

$$\mathcal{J}(\rho,\sigma) = S\left(\frac{\rho+\sigma}{2}\right) - \frac{1}{2}S(\rho) - \frac{1}{2}S(\sigma), \tag{301}$$

and using its square root as the distance measure of two states, i.e., $\mathcal{D} = \mathcal{J}^{1/2}$, they defined $C(\rho)$ of Eq. (143) as the total coherence, and

$$C_I(\rho) = \min_{\delta_S\in\mathcal{I}_S} \mathcal{D}(\rho,\delta_S), \tag{302}$$

as the intrinsic coherence which excludes the contribution of the subsystems, and

$$C_L(\rho) = \mathcal{D}(\delta_S^\star, \delta^\star), \tag{303}$$

as the local coherence. Here, $\mathcal{I}_S$ is comprised of the states $\delta_S = \sum_k p_k \tau_{k,1}^b \otimes \cdots \otimes \tau_{k,N}^b$ obtained by choosing all the possible basis set $\{|b_{l,n}\rangle\langle b_{l,n}|\}$ (with $\tau_{l,n}^b = \sum_l p_{l,n}|b_{l,n}\rangle\langle b_{l,n}|$), while $\delta^\star$ and $\delta_S^\star$ are the closest states for obtaining $C(\rho)$ and $C_I(\rho)$, respectively.

Building upon the above definitions, one has

$$C \leq C_L + C_I \leq \sum_n C_{L,n} + C_I, \tag{304}$$

where the first inequality is a direct result of the metric properties of $\mathcal{D}$ (i.e., the triangle inequality), and the second one is due to the subadditivity of C_L for product $\delta_S^\star$. In particular, for the N-partite system, one can divide it into different subsystems, and calculate the corresponding coherence. For example, we denote by $C_{1:2}$ and $C_{1:23}$ the intrinsic coherence between subsystems 1 and 2, and between subsystem 1 and the combined subsystem 23, and similarly for other cases. In this way, Radhakrishnan et al. (2016) defined the multipartite monogamy of coherence as

$$M = \sum_{n=2}^N C_{1:n} - C_{1:2\cdots N}, \tag{305}$$

which is monogamous for $M \leq 0$ and polygamous otherwise. They have calculated M for the three-qubit W and GHZ states, and showed the validity of it on analyzing coherence distribution in spin systems of the Heisenberg model.

Considering the fact that the coherence of a state ρ cannot be larger than the average coherence of its ensemble states $\{p_i, \rho_i\}$ (Baumgratz et al., 2014), Ma et al. (2017) studied the distribution of quantum coherence from another perspective. For $\rho = \sum_i p_i\rho_i$, they introduced a quantity which they called it accessible coherence,

$$C^{\mathrm{acc}}(\rho) = \sum_i p_i C(\rho_i) - C(\rho), \tag{306}$$

which characterizes the coherence one gains when knowing the information of the ensemble $\{p_i, \rho_i\}$. Moreover, if a maximization is taken over all state decompositions of $\rho = \sum_i p_i\rho_i$, one can obtain the maximal accessible coherence. In fact, the maximization is only necessary to be taken over the pure state decompositions of ρ due to compact convexity of the density matrix set.

For a bipartite state ρ^{AB}, they further defined the remaining coherence as

$$C^{\mathrm{rem}}(\rho^{AB}) = C(\rho^{AB}) - C(\rho^{A}) - C(\rho^{B}) - C^{\mathrm{acc}}(\rho^{A}) - C^{\mathrm{acc}}(\rho^{B}). \tag{307}$$

That is to say, the coherence in ρ^{AB} is divided into the local coherence and the local accessible coherence in its subsystems plus the remaining coherence. Through explicit examples, they showed that there are states for which the local coherence and local accessible coherence vanish, while the remaining coherence survives. Moreover, the remaining coherence can also be qualitatively different when being measured by the relative entropy and the l_1 norm, e.g., there are cases for which $C_{l_1}^{\mathrm{rem}}(\rho^{AB}) = 0$ and $C_r^{\mathrm{rem}}(\rho^{AB}) > 0$.

For the skew-information-based coherence measure of Eq. (226), Yu (2017) showed that for bipartite pure state $|\psi\rangle_{AB}$ with the reduced states ρ^A and ρ^B, the following polygamy relation holds

$$[1 - C_{sk}(\rho^A)][1 - C_{sk}(\rho^B)] \geq 1 - C_{sk}(|\psi\rangle_{AB}), \tag{308}$$

while for the mixed state ρ^{AB}, one has (Yu, 2017)

$$[1 - C_{sk}(\rho^A)][1 - C_{sk}(\rho^B)] \geq \sum_{kk'} \langle kk'|\rho^{AB}|kk'\rangle^2, \tag{309}$$

and

$$[1 - C_{sk}(\rho^A)][1 - C_{sk}(\rho^B)] \geq \frac{1}{c_s}[1 - C_{sk}(\rho^{AB})]^2, \tag{310}$$

where the right-hand side of Eq. (309) equals $\mathrm{tr}(\rho^{AB})^2 - C_{l_2}(\rho^{AB})$, and $c_s = [r - \sum_i C_{sk}(\rho^{Ai})][r - \sum_i C_{sk}(\rho^{Bi})]$, with $r = \mathrm{rank}(\rho^{AB})$, and ρ^{Ai} and ρ^{Bi} are the reduced states of the ith eigenstate of ρ^{AB}.

4.1.5. State ordering under different coherence measures

As various measures of quantum coherence have been put forward up to now, one may wonder whether they impose the same state ordering or not, just as the similar problem encountered when comparing various entanglement (Virmani and Plenio, 2000) and discordlike correlation measures (Hu and Fan, 2012b; Okrasa and Walczak, 2012).

When considering two measures of quantumness of a state denoted by Q_1 and Q_2, if

$$Q_1(\rho_1) \leq Q_1(\rho_2) \Longleftrightarrow Q_2(\rho_1) \leq Q_2(\rho_2), \tag{311}$$

for arbitrary two states ρ_1 and ρ_2, then they are said to give the same state ordering. Otherwise, they give inconsistent descriptions of quantumness.

By concentrating on the coherence measures, Liu et al. (2016a) examined state ordering problem imposed by the l_1 norm of coherence, relative entropy of coherence, and coherence of formation. Through explicit examples, they found that these measures also impose different orderings of states. In particular, for all measures of coherence that are equivalent for pure states, they must impose different orderings for general mixed states.

4.2. Complementarity of quantum coherence

As the measures of quantum coherence are basis dependent, a natural question that arises is how they behave when different bases are involved?

4.2.1. Mutually unbiased bases

For the l_1 norm and relative entropy measures of coherence, Cheng and Hall (2016) studied tradeoffs between coherence of the MUBs. Here, two observables and the resulting basis sets are said to be mutually unbiased if the measurement outcomes of either one with respect to any eigenstate of the other is uniformly distributed, i.e., the probability distribution is $\{1/d, \ldots, 1/d\}$ for a d-dimensional Hilbert space $\mathcal{H}$. For example, for qubits the three Pauli observables σ_x, σ_y, and σ_z are mutually unbiased. In fact, when $d = \dim \mathcal{H}$ is a prime power, there always exists a complete set of $d+1$ MUBs (Wootters, 1986; Wootters and Fields, 1989).

If one uses the l_1 norm of coherence $C_{l_1}(A_j, \rho)$ as a quantifier, with $\{A_j\}_{j=1}^{d+1}$ being the MUBs and $A_j = \{|a_i^{(j)}\rangle\}_{i=1}^{d}$, Cheng and Hall (2016) obtained

$$C_{l_1}(A_j, \rho) \leq \sqrt{d(d-1)[P(\rho) - P(A_j|\rho)]}, \tag{312}$$

where $P(\rho) = \mathrm{tr}\rho^2$ and $P(A_j|\rho) = \sum_i \langle a_i^{(j)}|\rho|a_i^{(j)}\rangle^2$ are called the quantum and classical purities, respectively. On the other hand, from the equality $\rho = \sum_j \rho(A_j) - \mathbb{1}$ (Ivanovic, 1981), with $\rho(A_j) = \Delta(A_j, \rho)$ denoting full dephasing of ρ in the basis

A_j [see Eq. (142)], one can prove $\sum_j P(A_j|\rho) = 1 + P(\rho)$, hence

$$\sum_{j=1}^{d+1} C_{l_1}^2(A_j, \rho) \le d(d-1)[dP(\rho) - 1]. \tag{313}$$

This is the complementarity relation for l_1 norm of coherence under the complete set of MUBs. It establishes connection between coherence and purity of a state, and bounds from above distribution of coherence as well. This bound is tight as it is saturated by the following states

$$\rho_\epsilon = \frac{\epsilon}{d-1}\mathbb{1} + \frac{d(1-\epsilon)-1}{d-1}|a_i^{(j)}\rangle\langle a_i^{(j)}|, \tag{314}$$

with $0 \le \epsilon \le 1$.

Similarly, Cheng and Hall (2016) derived a complementarity relation for the relative entropy of coherence

$$\begin{aligned}\sum_{j=1}^{d+1} C_r(A_j, \rho) \le &(d+1)[\log_2 d - S(\rho)] \\ &- \frac{(d-1)[dP(\rho)-1]}{d(d-2)}\log_2(d-1),\end{aligned} \tag{315}$$

this bound is saturated for the maximally coherent state $|\Psi_d\rangle$, and the second term on the right-hand side reduces to $[P(\rho) - 0.5]\log_2 e$ for $d = 2$.

Cheng and Hall (2016) also defined the mean coherence $\bar{C}(\rho)$ and the root mean square coherence rms$[C(\rho)]$ as

$$\begin{aligned}&\bar{C}(\rho) = \int dUC(U\rho U^\dagger, \rho), \\ &\mathrm{rms}[C(\rho)] = \left[\int dUC^2(U\rho U^\dagger, \rho)\right]^{1/2},\end{aligned} \tag{316}$$

where $\{U\}$ denote the unitaries which transform one basis set to another one, and dU is the normalized invariant Haar measure over $\{U\}$. Based on this, one can obtain

$$\begin{aligned}&\bar{C}_{l_1}(\rho) \le \mathrm{rms}[C_{l_1}(\rho)] \le \sqrt{\frac{d(d-1)[dP(\rho)-1]}{d+1}}, \\ &\bar{C}_r(\rho) = \sum_{n=2}^{d}\frac{1}{n}\log_2 e - [S(\rho) - Q(\rho)],\end{aligned} \tag{317}$$

where $\{\lambda_i\}_{i=1}^d$ are eigenvalues of ρ, and

$$Q(\rho) = -\sum_{i=1}^{d}\frac{\lambda_i^d \log_2\lambda_i}{\prod_{j\neq i}(\lambda_i - \lambda_j)}, \tag{318}$$

is the quantum subentropy (Datta et al., 2014).

4.2.2. Incompatible bases

Following the established notions for various entropic uncertainty relations (EURs) [see the review paper of Coles et al. (2015)], Singh et al. (2016a) further discussed tradeoff relations between quantum coherence of the MUBs.

First, for single-partite quantum state ρ, it follows immediately from the EUR $H(P) + H(Q) \ge \log_2(1/c) + S(\rho)$ that

$$C_r(Q, \rho) + C_r(R, \rho) \ge \log_2(1/c) - S(\rho), \tag{319}$$

where

$$c = \max_{k,l} |\langle\psi_k^Q|\psi_l^R\rangle|^2, \tag{320}$$

with $\{|\psi_k^Q\rangle\}$ and $\{|\psi_l^R\rangle\}$ denoting respectively, the eigenstates of the two incompatible observables Q and R. Similarly, by using the EUR derived by Korzekwa et al. (2014) and Sánchez-Ruiz (1998), one can obtain two new lower bounds for the sum of coherence, which are as follows

$$\begin{aligned}&C_r(Q, \rho) + C_r(R, \rho) \ge \log_2(1/c)[1 - S(\rho)], \\ &C_r(Q, \rho) + C_r(R, \rho) \ge H\left(\frac{1+\sqrt{2c-1}}{2}\right) - 2S(\rho),\end{aligned} \tag{321}$$

and they may be more or less optimal than the bound of Eq. (319) due to the different values of c and the form of ρ.

For single-qubit state ρ, Yuan et al. (2017) obtained new lower bounds for the sum of coherence measures under two incompatible bases. By denoting $P' = 2\mathrm{tr}\rho^2 - 1$, these bounds can be written explicitly as

$$\begin{aligned} &C_r(Q,\rho) + C_r(R,\rho) \geq H\left(1 + \frac{\sqrt{P'}(2\sqrt{c}-1)}{2}\right) - S(\rho), \\ &C_{l_1}(Q,\rho) + C_{l_1}(R,\rho) \geq 2\sqrt{P'c(1-c)}, \\ &R_I(Q,\rho) + R_I(R,\rho) \geq H\left(\frac{1+\sqrt{1-4P'(\sqrt{c}-c)}}{2}\right). \end{aligned} \tag{322}$$

Second, for the bipartite state ρ^{AB}, by using the quantum-memory-assisted EUR (Berta et al., 2010)

$$S(Q|B) + S(R|B) \geq \log_2(1/c) + S(A|B), \tag{323}$$

and taking the eigenstates $\Xi = \{|\psi_k^X\rangle|\varphi_j^B\rangle\}$ ($X = \{Q, R\}$, and $|\varphi_j^B\rangle$ is the eigenstate of $\rho^B = \mathrm{tr}_A\rho^{AB}$) as the basis, we have

$$C_r(\Xi, \rho^{AB}) + C_r(\Xi, \rho^{AB}) \geq \log_2(1/c) - S(A|B), \tag{324}$$

where the bound on the right-hand side can be further tightened by using the concept of QD (Pati et al., 2012).

Similarly, if one considers the mutually incompatible observables $\{Q_1, Q_2, \ldots, Q_n\}$, with the corresponding eigenstate bases $\{|\psi_{k_1}^{Q_1}\rangle|\psi_{k_2}^{Q_2}\rangle\cdots|\psi_{k_n}^{Q_n}\rangle\}$ and $\Xi_i = \{|\psi_{k_i}^{Q_i}\rangle|\varphi_j^B\rangle\}$, then by using the formulas established by Liu and Mu (2015a), one can obtain

$$\begin{aligned} &\sum_{i=1}^{n} C_r(Q_i, \rho) \geq \log_2(1/b) - S(\rho), \\ &\sum_{i=1}^{n} C_r(\Xi_i, \rho^{AB}) \geq \log_2(1/b) - S(A|B), \end{aligned} \tag{325}$$

and this can be considered as an extension of the results for two mutually unbiased observables.

4.2.3. Complementarity between coherence and mixedness

For the class of states with fixed mixedness, the amount of quantum coherence contained in them may be different. Using the linear entropy measure of mixedness

$$M_l(\rho) = \frac{d}{d-1}(1 - \mathrm{tr}\rho^2), \tag{326}$$

and the l_1 norm of coherence given in Eq. (157), Singh et al. (2015) derived a tradeoff relation between the two quantities,

$$\frac{C_{l_1}^2(\rho)}{(d-1)^2} + M_l(\rho) \leq 1, \tag{327}$$

where the first term on the left-hand side can be seen as the square of the normalized coherence, $\tilde{C}_{l_1}(\rho) := C_{l_1}(\rho)/(d-1)$. It sets a fundamental limit to the amount of coherence that can be extracted from the class of states with equal mixedness, and vice versa.

Moreover, in the same vein with the definition of maximally entangled mixed states (Ishizaka and Hiroshima, 2000; Peters et al., 2004; Verstraete and Audenaert, 2001), Singh et al. (2015) considered the class of MCMS given in Eq. (168), and found that the upper bound in Eq. (328) is saturated, as it gives

$$C_{l_1}(\rho_{\mathrm{mcms}}) = p(d-1),\ M_l(\rho_{\mathrm{mcms}}) = 1 - p^2, \tag{328}$$

for $1 \leq p \leq 1$. In fact, ρ_{mcms} constitutes also the class of states with maximal mixedness for fixed coherence.

Although Singh et al. (2015) pointed out that similar tradeoffs apply to the relative entropy of coherence [i.e., $C_r(\rho) + S(\rho) \leq 1$, which is incorrect as $C_r(\rho) + S(\rho) = S(\rho_{\mathrm{diag}}) \leq \log_2 d$], and the fidelity-based measure of coherence for single qubit state [see Eq. (174)], it remains as a challenge to generalize the complementarity relation (328) to other coherence measures which are on equal footing with the l_1 norm of coherence. Some other progress have been made, e.g., Zhang et al. (2017a) have showed that if the mixedness of ρ is defined via the fidelity as

$$M_g(\rho) = F(\rho, \mathbb{1}/d) = \frac{1}{d}(\mathrm{tr}\sqrt{\rho})^2, \tag{329}$$

then by combining this with Eq. (178) and further using the mean inequality $\sum_i x_i \le (d\sum_i x_i^2)^{1/2}$ ($\forall x_i \in \mathbb{R}$), it is easy to see that

$$C_g(\rho) + M_g(\rho) \le 1 - \sum_i b_{ii}^2 + \frac{1}{d}\left(\sum_i b_{ii}\right)^2 \le 1, \tag{330}$$

with equality holding for $\rho_{\rm mcms}$ of Eq. (168).

Stimulated by the work of Horodecki et al. (2003a) and Gour et al. (2015), a resource theory of purity was established by Streltsov et al. (2018). In this framework, the only free state is the maximally mixed state $\rho_{\rm mm} = \mathbb{1}/d$, and the set of free operations is the unital operations $\Lambda_{\rm U}$. A functional $P(\rho)$ is said to be a purity monotone if it is nonnegative and $P(\Lambda_{\rm U}[\rho]) \le P(\rho)$. $P(\rho)$ is a purity measure if it further satisfies the additivity property $P(\rho\otimes\sigma) = P(\rho) + P(\sigma)$ and normalization condition $P(|\psi\rangle) = \log_2 d$. Moreover, it is convex if $\sum_i p_i P(\rho_i) \ge P(\sum_i p_i\rho_i)$.

Based on the above framework, Streltsov et al. (2018) introduced a coherence-based purity monotone

$$P_C(\rho) := \max \Lambda_{\rm U} C(\Lambda_{\rm U}[\rho]) = C(\rho_{\rm max}), \tag{331}$$

with C being any MIO monotone. When C is defined by the contractive distance $\mathcal{D}$, the combination of the above equation with Eq. (253) further gives $P_C(\rho) = \mathcal{D}(\rho, \mathbb{1}/d)$. This shows another connection between purity of a state and the maximum amount of quantum coherence achievable by suitable unitary operation.

Moreover, Streltsov et al. (2018) also introduced a Rényi α-entropy purity measure

$$P_\alpha(\rho) = \log_2 d - \frac{1}{1-\alpha}\log_2\{{\rm tr}(\rho^\alpha)\}, \tag{332}$$

for $\alpha \ge 0$, which is also convex when $\alpha \in [0, 1]$. In particular, the Rényi 2-entropy purity measure $P_2(\rho) = \log_2\{d{\rm tr}(\rho^2)\}$ is quantitatively related to the linear entropy of purity ${\rm tr}\rho^2$, and when $\alpha \to 1$, we have the traditional relative entropy of purity

$$P_r(\rho) = \log_2 d - S(\rho). \tag{333}$$

4.3. *Duality of coherence and path distinguishability*

With roots in quantum optics, quantum coherence lies at the heart of interference phenomenon. The presence of coherence in a quantum system can be seen as a manifestation of the wave nature of it (Bagan et al., 2016; Bera et al., 2015), while the path distinguishability or which-path information signifies its complementarity aspect, i.e., the particle nature of it. The quantitative connections between them can be investigated in the context of unambiguous quantum state discrimination (UQSD) or ambiguous quantum state discrimination (AQSD), which are implementable in interference experiments.

4.3.1. *Unambiguous quantum state discrimination*

Bera et al. (2015) proposed to use quantum coherence to signify the wave nature of a particle, and the upper bound of the success probability of UQSD to signify its particle aspect. Let $\{|\xi_i\rangle\}$ be a collection of states which may be nonorthogonal, then the task of UQSD is to find with certainty which of them is the given one, see Qiu (2002); Zhang et al. (2001) and references therein.

In the N-path interference experiment, we denote by $\{|\psi_i\rangle\}$ the orthogonal basis state of the path. Then if the initial state of the particle entering the interferometer is

$$|\psi\rangle_s = \sum_{i=1}^{N} c_i|\psi_i\rangle, \tag{334}$$

with $\sum_i |c_i|^2 = 1$, and the related detector state is $|0_i\rangle$. Their combined state after the interaction operation is

$$|\psi\rangle_{sd} = \sum_{i=1}^{N} c_i|\psi_i\rangle\otimes|\xi_i\rangle. \tag{335}$$

To discriminate the which-path information, the experimenter can perform measurements on the detector states. The probability for successfully discriminating them is proved to be bounded from above by (Qiu, 2002; Zhang et al., 2001)

$$p_{\rm uqsd} \le \mathcal{D}_Q := 1 - \frac{1}{N-1}\sum_{i\ne j}|c_i c_j||\langle\xi_i|\xi_j\rangle|, \tag{336}$$

where $\mathcal{D}_Q$ sets a limit to the ability of the experimenter to distinguish between the states $\{|\xi_i\rangle\}$ (and hence $\{|\psi_i\rangle\}$), although it may not be achievable in real experiments.

On the other hand, the postmeasurement state of the particle is

$$\rho_s' = \sum_{i,j} c_i c_j^* \langle \xi_j | \xi_i \rangle |\psi_i\rangle\langle\psi_j|, \tag{337}$$

hence

$$C_{l_1}(\rho_s') = \sum_{i\neq j} |c_i c_j| |\langle \xi_j | \xi_i \rangle|, \tag{338}$$

in the path basis $\{|\psi_i\rangle\}$. Based on these, Bera et al. (2015) derived the following relation

$$\frac{C_{l_1}(\rho_s')}{N-1} + \mathcal{D}_Q = 1, \tag{339}$$

It characterizes in a quantitative way the wave–particle duality. In particular, for two- and three-path situations with uniform $|c_i|$, the normalized coherence $\tilde{C}_{l_1} = C_{l_1}/(N-1)$ is also quantitatively related to the interference fringe visibility $\mathcal{V}$, namely, $\tilde{C}_{l_1} = \mathcal{V}$ and $\tilde{C}_{l_1} = 2\mathcal{V}/(3-\mathcal{V})$, respectively (Bera et al., 2015).

Moreover, for the case of initially mixed particle state $\rho_s = \sum_i \rho_{ij} |\psi_i\rangle\langle\psi_j|$ and pure detector state, or the more general case of both initially mixed particle and detector states, Bera et al. (2015) showed that the equality of Eq. (338) becomes inequality

$$\frac{C_{l_1}(\rho_s')}{N-1} + \mathcal{D}_Q \leq 1. \tag{340}$$

By using a slightly different path distinguishability $\mathcal{D} = [\mathcal{D}_Q(2-\mathcal{D}_Q)]^{1/2}$, Qureshi and Siddiqui (2017) further gave an equivalent form of Eq. (340), i.e.,

$$\frac{C_{l_1}^2(\rho_s')}{(N-1)^2} + \mathcal{D}^2 \leq 1, \tag{341}$$

which is similar to the complementarity relation $\mathcal{D}^2 + \mathcal{V}^2 \leq 1$ given by Englert (1996).

If we put a screen behind the slits, the interference pattern of the particle is described by the probability density of particle hitting the screen at particular position. Paul and Qureshi (2017) considered one such kind of problem. For the particle-detector state $|\psi\rangle_{sd}$ of Eq. (335), the expression for the pattern on the screen is given by $|\langle x|\psi\rangle_{sd}|^2$. For an N-slit experiment with the width of the slits being ϵ, and the distance between any two neighboring slits (between the slits and the screen) is ℓ (D), then if we assume that the state that emerges from the jth slit is a Gaussian along the x axis and centered at $x_j = j\ell$, the state of the particle hitting the screen at a position x will be (Paul and Qureshi, 2017)

$$\langle x|\psi(t)\rangle_{sd} = A_t \sum_{j=1}^{N} c_j \exp\left[-\frac{(x-j\ell)^2}{\epsilon^2 + i\lambda D/\pi}\right] |\xi_j\rangle, \tag{342}$$

where $A_t = \{2/[\pi(\epsilon + i\lambda D/\pi\epsilon)]\}^{1/4}$ (i is the imaginary unit), and $|\psi(t)\rangle_{sd}$ is the evolved state with $|\psi(0)\rangle_{sd}$ being given by Eq. (335).

Then by using the facts that $\epsilon^2 \ll (\lambda D/\pi)^2$ and the distance between the primary maxima $\lambda D/\ell \gg j\ell$, one can obtain that the intensity of the fringe $I(x) = |\langle x|\psi(t)\rangle_{sd}|^2$ at position x is given by Paul and Qureshi (2017)

$$\begin{aligned} I(x) =& |A_t|^2 \exp\left[-\frac{2\epsilon^2 x^2}{(\lambda D/\pi)^2}\right] \Bigg\{ 1 + \sum_{j\neq k} |c_j c_k| \\ & \times |\langle \xi_j|\xi_k\rangle| \cos\left[\frac{2\pi x\ell(k-j)}{\lambda D} + \theta_k - \theta_j\right] \Bigg\}, \end{aligned} \tag{343}$$

where we have defined $c_k|\xi_k\rangle = |c_k||\xi_k'\rangle e^{i\theta_k}$, with $|\xi_k'\rangle$ being real.

By choosing $\theta_k = \theta_j$ $(\forall k, j)$, one can obtain from the above equation that at positions $x_m = m\lambda D/\ell$ $(m \in \mathbb{Z})$ of the primary maxima, the intensity of the fringe is given by

$$I_{\max} = |A_t|^2 \exp\left[-\frac{2\epsilon^2 x_m^2}{(\lambda D/\pi)^2}\right] \left(1 + \sum_{j\neq k} |c_j c_k| |\langle \xi_j|\xi_k\rangle|\right). \tag{344}$$

Moreover, when a phase randomizer is applied to the setup such that the phases of the incoming state at different slits are randomized (i.e., the incoming state becomes incoherent), the cosine term of Eq. (343) will disappear, thus we have

$$I_{\rm inc} = |A_t|^2 \exp\left[-\frac{2\epsilon^2 x_m^2}{(\lambda D/\pi)^2}\right]. \tag{345}$$

Finally, by combining the above two results with Eq. (338), one can obtain directly that

$$\frac{I_{\max}-I_{\mathrm{inc}}}{I_{\mathrm{inc}}}=C_{l_1}(\rho_s'), \tag{346}$$

In fact, from Eq. (343) one can see directly that when the which-path information is completely indistinguishable (i.e., $\langle\xi_j|\xi_k\rangle=1, \forall k,j$), the intensity of the interference fringe at the primary maximum $x_m=m\lambda D/\ell$ is given by

$$I_{\max}^{\parallel}=|A_t|^2\exp\left[-\frac{2\epsilon^2x_m^2}{(\lambda D/\pi)^2}\right]\left(1+\sum_{j\neq k}|c_jc_k|\right). \tag{347}$$

Similarly, when the which-path information is completely distinguishable (i.e., $\langle\xi_j|\xi_k\rangle=0, \forall k,j$), the intensity of the interference fringe at the primary maximum $x_m=m\lambda D/\ell$ turns out to be

$$I_{\max}^{\perp}=|A_t|^2\exp\left[-\frac{2\epsilon^2x_m^2}{(\lambda D/\pi)^2}\right]. \tag{348}$$

Then it is obvious that

$$\frac{I_{\max}^{\parallel}-I_{\max}^{\perp}}{I_{\max}^{\perp}}=C_{l_1}(\rho_s'). \tag{349}$$

The implementation of the above scheme for measuring quantum coherence depends essentially on whether there exists such a path detector which is (at least) capable of making the which-path information completely indistinguishable and distinguishable.

4.3.2. Ambiguous quantum state discrimination

Different from UQSD, one always has a result in the AQSD experiments, but it may be right or wrong, and the task of the experimenter is to minimize the probability of being wrong to its theoretical limit (Englert, 1996), hence it is also known as minimum-error state discrimination.

By using the l_1 norm of coherence to characterize the wave nature, and an upper bound on the average probability p_{aqsd} of successfully discriminating the path information to characterize the particle nature, Bagan et al. (2016) derived the following duality relation between them

$$\left(\frac{C_{l_1}(\rho_s')}{N-1}\right)^2+\left(\frac{Np_{\mathrm{aqsd}}-1}{N-1}\right)^2\leq 1, \tag{350}$$

where ρ_s' is the same as Eq. (337). To discriminate the which-path information, the experimenter can perform POVM on the reduced detector state $\rho_d=\sum_i|c_i|^2\rho_i^{\xi}$ (with $\rho_i^{\xi}=|\xi_i\rangle\langle\xi_i|$), and the average probability $p_{\mathrm{aqsd}}=\sum_i|c_i|^2\mathrm{tr}(\Pi_i|\xi_i\rangle\langle\xi_i|)$ is shown to be upper bounded by

$$p_{\mathrm{aqsd}}\leq\frac{1}{N}+\frac{1}{2N}\sum_{ij}\|\Lambda_{ij}\|_1, \tag{351}$$

where $\{\Pi_i\}$ is the set of POVM, while $\Lambda_{ij}=|c_i|^2\rho_i^{\xi}-|c_j|^2\rho_j^{\xi}$ is the Helstron matrix of the state pair $(\rho_i^{\xi},\rho_j^{\xi})$, and

$$\|\Lambda_{ij}\|_1=2\sqrt{\left(\frac{|c_i|^2+|c_j|^2}{2}\right)^2-|c_ic_j|^2|\langle\xi_i|\xi_j\rangle|^2}. \tag{352}$$

Bagan et al. (2016) also derived a duality relation between the relative entropy of coherence and path distinguishability, i.e.,

$$C_r(\rho_s')+H(M:D)\leq H(\{p_i\}), \tag{353}$$

where $H(M:D)=H(\{p_j\})+H(\{q_i\})-H(\{p_{ij}\})$ is the mutual information of $D=\{p_j\}$ and $M=\{q_i\}$, and $H(\cdot)$ is the Shannon entropy, with the probabilities $p_{ij}=\mathrm{tr}(\Pi_i\rho_j^{\xi})p_j$, $p_j=\sum_i p_{ij}=|c_j|^2$, and $q_i=\sum_j p_{ij}=\mathrm{tr}(\Pi_i\rho_d)$.

Eq. (353) holds as well even if its second term on the left-hand side is replaced by the accessible information I_{acc}, which is defined as the maximum value of $H(M:D)$ over all possible POVMs, and characterizes how well an experimenter can do at inferring the detector states.

Recently, it has been pointed out by Qureshi and Siddiqui (2017) that the upper bound of Eq. (350) may not be saturated for general pure states $|\psi\rangle_{sd}$ if $N\geq 3$. It is also doubted as the two terms on its left-hand side (which characterize the wave and particle nature of a quanton, respectively) can increase or decrease simultaneously.

4.4. Distillation and dilution of quantum coherence

4.4.1. Standard coherence distillation and dilution

In the same spirit as entanglement distillation and entanglement formation, one can also consider the tasks of coherence distillation and coherence formation by incoherent operations Λ, see Fig. 5(a). The former corresponds to the transformation of a general state ρ to the maximally coherent one, i.e., the distillation of ρ to $\Psi_2 = |\Psi_2\rangle\langle\Psi_2|$, while the latter is the formation of ρ from Ψ_2. The corresponding optimal rate can be recognized as an operational measure of coherence.

In the asymptotic setting (i.e., infinitely many copies of ρ), Winter and Yang (2016) defined the distillable coherence as the maximal rate at which Ψ_2 can be obtained from ρ, i.e.,

$$C_d(\rho) = \sup\{R : \lim_{\substack{n\to\infty \\ \varepsilon\to 0}} (\inf_{\Lambda} \|\Lambda(\rho^{\otimes n}) - \Psi_2^{\otimes \lfloor nR \rfloor}\|_1) \leq \varepsilon\}, \tag{354}$$

and the coherence cost which is the minimal rate at which Ψ_2 has to be consumed for formatting ρ is dually to Eq. (354), with only the supremum being replaced by the infimum, and Λ acting on $\Psi_2^{\otimes \lceil nR \rceil}$. The central results are that $C_d(\rho)$ equals the relative entropy of coherence $C_r(\rho)$, while $C_c(\rho)$ equals the coherence of formation

$$C_f(\rho) := \min_{\{p_i,\psi_i\}} \sum_i p_i S(\Delta[\psi_i]), \tag{355}$$

which involves a minimization over all pure state decompositions of ρ showed in Eq. (13) (Aberg, 2006), and the additivity of C_f and C_r implies that both C_d and C_c are additive as well.

Moreover, different from the possible bound entanglement in a state, Winter and Yang (2016) found that there is no bound coherence, that is, there does not exist quantum state for which its creation consumes coherence while no coherence could be distilled from it. Thus,

$$C_d(\rho) = 0 \Rightarrow C_c(\rho) = 0, \tag{356}$$

which reveals that quantum coherence in any state is always distillable.

Bu et al. (2017b) considered an one-shot version of coherence distillation. They defined the relevant coherence cost for formatting a quantum state ρ under MIO as

$$C_{c,\mathrm{MIO}}^{(1),\varepsilon}(\rho) = \inf_{\substack{\{\Lambda_{\mathrm{MIO}}\} \\ M\in\mathbb{Z}}} \{\log_2 M | F[\rho, \Lambda_{\mathrm{MIO}}(\Psi_+^M)] \geq 1-\varepsilon\}, \tag{357}$$

where $\Psi_+^M = |\Psi_+^M\rangle\langle\Psi_+^M|$ with $|\Psi_+^M\rangle = \sum_{i=1}^M |i\rangle/\sqrt{M}$, and $F(\cdot)$ is the Uhlmann fidelity given in Eq. (52) [note that it equals the square of that adopted by Bu et al. (2017b)]. Then, they showed that, for any $\varepsilon > 0$, one has

$$C_{\max}^{2\sqrt{\varepsilon}}(\rho) \leq C_{c,\mathrm{MIO}}^{(1),\varepsilon}(\rho), \tag{358}$$

so the smooth maximum relative entropy of coherence bounds from below the one-shot coherence cost.

Similar to the one-shot coherence distillation, one can also consider the one-shot version of coherence dilution, in which the corresponding coherence cost reads

$$C_{\mathcal{O}}^{\varepsilon}(\rho) = \inf_{\Lambda\in\mathcal{O}} \{\log_2 M | F[\rho, \Lambda(\Psi_+^M)] \geq 1-\varepsilon\}, \tag{359}$$

where $\mathcal{O}$ is one of the free operations {MIO,DIO,IO,SIO}. To establish an operational interpretation for the coherence measure, Zhao et al. (2018) further introduced an ε-smoothed coherence measure $C^{\varepsilon}(\rho) = \min_{\rho'\in B_{\varepsilon}(\rho)} C(\rho')$, where $B_{\varepsilon}(\rho) = \{\rho' | F(\rho, \rho' \geq 1-\varepsilon)\}$. Based on these preliminaries, they proved that for any $\varepsilon > 0$, we have

$$\begin{aligned} &C_{\max}^{\varepsilon}(\rho) \leq C_{\mathrm{MIO}}^{\varepsilon}(\rho) \leq C_{\max}^{\varepsilon}(\rho) + 1, \\ &C_{\Delta,\max}^{\varepsilon}(\rho) \leq C_{\mathrm{DIO}}^{\varepsilon}(\rho) \leq C_{\Delta,\max}^{\varepsilon}(\rho) + 1, \\ &C_{\mathrm{MIO}}^{\varepsilon}(\rho) = C_{\mathrm{SIO}}^{\varepsilon}(\rho) = C_0^{\varepsilon}(\rho), \end{aligned} \tag{360}$$

and in the asymptotic limit, the ε-smoothed coherence equivalents either to the relative entropy of coherence (Baumgratz et al., 2014) or to the coherence of formation (Winter and Yang, 2016), i.e.,

$$\begin{aligned} &C_{\mathrm{MIO}}^{\infty}(\rho) = C_{\mathrm{DIO}}^{\infty}(\rho) = C_r(\rho), \\ &C_{\mathrm{IO}}^{\infty}(\rho) = C_{\mathrm{SIO}}^{\infty}(\rho) = C_f(\rho). \end{aligned} \tag{361}$$

where $C_{\mathcal{O}}^{\infty}(\rho) = \lim_{\varepsilon\to 0, n\to\infty} C_{\mathcal{O}}^{\varepsilon}(\rho^{\otimes n})/n$. This result, together with that of Winter and Yang (2016), implies that the role of MIO and IO in the asymptotic scenario of coherence dilution is the same as we also have $C_{\mathrm{IO}}^{\infty}(\rho) = C_r(\rho)$.

Please cite this article in press as: Hu M.-L., et al., Quantum coherence and geometric quantum discord. Physics Reports (2018), https://doi.org/10.1016/j.physrep.2018.07.004.

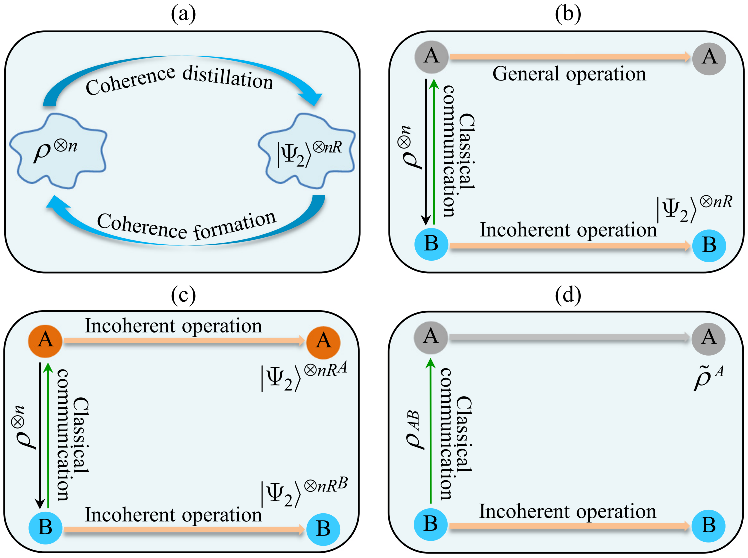

Fig. 5. (a) Coherence distillation and coherence formation. (b) Assisted coherence distillation with general operation on party A, incoherent operation on B, and two-way (one-way) classical communication between them. (c) Coherence distillation with incoherent operations on both A and B and classical communication. (d) Remote creation of coherence on party A with quantum operation on B and classical communication from B to A.

4.4.2. Assisted coherence distillation

In analogy to assisted entanglement distillation, Chitambar et al. (2016) investigated the task of assisted coherence distillation in the setting of local quantum-incoherent operations and classical communication (LQICC), see Fig. 5(b). In this task, two players, Alice and Bob, share n copies of ρ^{AB}, and Alice's objective is to help Bob to distill as much quantum coherence as possible. Different from the allowable LOCC in assisted entanglement distillation, LQICC represents quantum operations Λ_{QI} that are general on Alice's side and incoherent on Bob's side. In this setting, the set $\mathcal{QI}$ of free states called the quantum-incoherent (QI) states are given by $\chi^{AB} = \sum_i p_i \rho_i^A \otimes |i^B\rangle\langle i^B|$, with p_i the probabilities, ρ_i^A arbitrary states for subsystem A, and $\{|i^B\rangle\}$ the incoherent basis for subsystem B.

Formally, the generated maximum coherence is called "coherence of collaboration" (CoC) for two-way communication, and "coherence of assistance" (CoA) in the one-way situation, for which Alice holds a purifying state and only she is allowed to announce the measurement results. Chitambar et al. (2016) defined the optimal rate of distillable CoC as

$$C_d^{A|B}(\rho) = \sup\{R : \lim_{n\to\infty} (\inf_{\Lambda_{QI}} \|\Lambda_{QI}(\rho^{\otimes n}) - \Psi_2^{\otimes \lfloor nR \rfloor}\|_1) = 0\}, \tag{362}$$

where $C_d^{A|B}$ is upper bounded by the QI relative entropy $C_r^{A|B}$, i.e.,

$$C_d^{A|B}(\rho^{AB}) \le C_r^{A|B}(\rho^{AB}), \tag{363}$$

with equality holding for any pure state, and

$$C_r^{A|B}(\rho^{AB}) = \min_{\chi^{AB}\in\mathcal{QI}} S(\rho^{AB} \parallel \chi^{AB}) = S(\Delta^B[\rho^{AB}]) - S(\rho^{AB}), \tag{364}$$

where $\Delta^B[\rho^{AB}]$ denotes dephasing of ρ^{AB} in the incoherent basis of B.

Moreover, when extended to the situation with $N \ge 2$ assistants, the global operations across all auxiliary systems do not necessarily outperform the local operations on generating coherence, e.g., for the initial state $|\Psi\rangle^{A_1\cdots A_N B}$ with B being a qubit, local operations on $A_1, \ldots, A_N$ together with classical communication are enough to localize maximum coherence on B.

Chitambar et al. (2016) also proposed quantitative definitions of CoA and the regularized CoA, which are given respectively, by

$$C_a(\rho) = \max_{\{p_i,\psi_i\}} \sum_i p_i C_r(\psi_i), \;\; C_a^\infty(\rho) = \lim_{n\to\infty} \frac{1}{n} C_a(\rho^{\otimes n}), \tag{365}$$

and the maximization is taken over the pure state decompositions ρ showed in Eq. (13). Moreover, for qubit states ρ, the CoA is shown to be additive, i.e., $C_a(\rho^{\otimes n}) = nC_a(\rho)$.

Notably, there exists some resemblance between CoA of the state $\rho = \sum_{i,j}\rho_{ij}|i\rangle\langle j|$ and entanglement of assistance (EoA) of the related maximally correlated state ρ_{mc} of Eq. (290), that is,

$$C_a(\rho) = E_a(\rho_{\mathrm{mc}}),\ C_a^{\infty}(\rho) = E_a^{\infty}(\rho_{\mathrm{mc}}) = S(\Delta[\rho]). \tag{366}$$

Moreover, for pure state $|\psi\rangle^{AB}$, the CoC equals the regularized CoA, i.e.,

$$C_d^{A|B}(|\psi\rangle^{AB}) = C_a^{\infty}(\rho^B) = S(\Delta[\rho^B]), \tag{367}$$

which immediately yields that the maximum extra coherence that Bob can gain [compared with the standard distillation protocol (Winter and Yang, 2016)] with Alice's assistance equals the von Neumann entropy of ρ^B.

By replacing LQICC with the local incoherent operations and classical communication (LIOCC), Chitambar and Hsieh (2016) further studied the coherence–entanglement tradeoffs in a task similar to Chitambar et al. (2016), but now both the two parties' operations are restricted to be local incoherent, see Fig. 5(c). In this new setting, if we denote by R^A (R^B) the rate of coherence formation for Alice (Bob), and E^{co} that of entanglement formation between Alice and Bob, then the triple (R^A, R^B, E^{co}) is achievable if for every $\varepsilon > 0$ there exists a LIOCC Λ_{II} and integer n such that

$$\|\Lambda_{II}(\Psi_2^{\otimes\lceil n(R^A+\varepsilon)\rceil} \otimes \Psi_2^{\otimes\lceil n(R^B+\varepsilon)\rceil} \otimes \Phi_2^{\otimes\lceil n(E^{co}+\varepsilon)\rceil}) - \rho^{\otimes n}\|_1 \le \varepsilon, \tag{368}$$

where the two Ψ_2 belong to Alice and Bob, respectively, while $\Phi_2 = |\Phi_2\rangle\langle\Phi_2|$ with $|\Phi_2\rangle = (|00\rangle + |11\rangle)/\sqrt{2}$ is shared between them. Similarly, (R^A, A^B, E^{co}) is the achievable coherence–entanglement distillation triple if

$$\|\Lambda_{II}(\rho^{\otimes n}) - \Psi_2^{\otimes\lfloor n(R^A-\varepsilon)\rfloor} \otimes \Psi_2^{\otimes\lfloor n(R^B-\varepsilon)\rfloor} \otimes \Phi_2^{\otimes\lfloor n(E^{co}-\varepsilon)\rfloor}\|_1 \le \varepsilon. \tag{369}$$

For pure states $\Psi = |\Psi\rangle^{AB}\langle\Psi|$, Chitambar and Hsieh (2016) obtained the possible optimal triples of resource formation

$$\begin{aligned}
&(R^A, A^B, E^{co}) = (0, S(B|A)_{\Delta(\Psi)}, S(A)_{\Delta(\Psi)}),\\
&(R^A, A^B, E^{co}) = (S(A)_{\Delta(\Psi)}, S(B|A)_{\Delta(\Psi)}, E(\Psi)),\\
&(R^A, A^B, E^{co}) = (0, 0, S(AB)_{\Delta(\Psi)}),
\end{aligned} \tag{370}$$

with $S(X)_{\Delta(\Psi)}$ $[S(X|Y)_{\Delta(\Psi)}]$ the von Neumann entropy (conditional entropy) of $\Delta(\Psi)$, and $E(\Psi) = S(A)_\Psi$ the entanglement of Ψ. Using monotonicity of a LIOCC monotone

$$C_L(\rho^{AB}) = \min_{\{p_i,\psi_i\}} \sum_i p_i C_L(\psi_i), \tag{371}$$

with $\rho^{AB} = \sum_i p_i\psi_i$, $\psi_i = |\psi_i\rangle\langle\psi_i|$, and

$$C_L(\psi_i) = S(A)_{\Delta(\psi_i)} + S(B)_{\Delta(\psi_i)} - E(\psi_i), \tag{372}$$

Chitambar and Hsieh (2016) also derived the optimal resource distillation triples

$$\begin{aligned}
&(R^A, A^B, E^{co}) = (S(A)_{\Delta(\Psi)} - E(\Psi), S(B)_{\Delta(\Psi)}, 0),\\
&(R^A, A^B, E^{co}) = (0, S(B|A)_{\Delta(\Psi)}, I(A\!:\!B)_{\Delta(\Psi)}),
\end{aligned} \tag{373}$$

where $I(A\!:\!B)_{\Delta(\Psi)}$ is the mutual information of $\Delta(\Psi)$.

It is evident that the distillable coherence rate sum $C_D^{\mathrm{LIOCC}} = R^A + R^B$ that can be distilled simultaneously at Alice and Bob's side is constrained by their shared entanglement. By further defining two similar quantities C_D^{Global} and C_D^{LIO}, the former with global incoherent operations, and the latter with local incoherent operations without classical communication, Chitambar and Hsieh (2016) found that for Ψ, the differences

$$\begin{aligned}
&\delta(\Psi) = C_D^{\mathrm{Global}}(\Psi) - C_D^{\mathrm{LIOCC}}(\Psi),\\
&\delta_c(\Psi) = C_D^{\mathrm{LIOCC}}(\Psi) - C_D^{\mathrm{LIO}}(\Psi),
\end{aligned} \tag{374}$$

are given by

$$\delta(\Psi) = E(\Psi) - I(A\!:\!B)_{\Delta(\Psi)},\quad \delta_c(\Psi) = E(\Psi). \tag{375}$$

They describe, respectively, the extra coherence rates that can only be distilled by nonlocal incoherent operations and by using the data communicated via a classical channel.

Ma et al. (2016b) considered a similar scenario of collaborative creation of coherence, see Fig. 5(d). Here, two parties share a state ρ^{AB}, and their aim is to create coherence on A with the help of quantum operation solely on B and one-way classical communication from B to A. They called this as remote creation of coherence (RCC), and obtained relations between the created coherence and entanglement of ρ^{AB}. By using the operator-sum representation $\mathcal{E}(\cdot) = \sum_i E_i(\cdot)E_i^\dagger$, and denoting $\tilde{\rho}^A = \mathrm{tr}_B(\mathbb{1}_A \otimes \mathcal{E})\rho^{AB}/p'$ (with $p' = \mathrm{tr}(\mathbb{1}_A \otimes \mathcal{E})\rho^{AB}$ the probability of getting $\tilde{\rho}^A$), they proved that the RCC $C(\tilde{\rho}^A) = 0$ if and only if $\rho^{AB} = \sum_i p_i \sum_k q_k^i |k\rangle\langle k| \otimes \rho_i^B$, namely, it is an incoherent-quantum state.

For the initial pure state $|\psi^{AB}\rangle$ with vanishing coherence on A, the RCC is nonzero if and only if there exists a basis $\{|i\rangle\}$ which gives $[N, (\langle i| \otimes \mathbb{1})|\psi^{AB}\rangle\langle\psi^{AB}|(|i\rangle \otimes \mathbb{1})] \neq 0$, with $N = \sum_i E_i^\dagger E_i \leq \mathbb{1}$. The amount of RCC measured by the l_1 norm is bounded above by

$$C_{l_1}(\tilde{\rho}^A) \leq \frac{E(|\psi^{AB}\rangle)}{p'} \sqrt{\sum_{j<i} |N_{ji}|^2}, \tag{376}$$

where $E(|\psi^{AB}\rangle)$ denotes the concurrence of $|\psi^{AB}\rangle$, and N_{ji}s are matrix elements of N under the Schmidt decomposition basis of $\rho^B = \mathrm{tr}_A(|\psi^{AB}\rangle\langle\psi^{AB}|)$. Furthermore, if the channel $\mathcal{E}$ is trace preserving, the average RCC

$$\bar{C}_{l_1}(|\psi^{AB}\rangle) := \sum_i p_i C_{l_1}(\tilde{\rho}_i^A), \tag{377}$$

with $\tilde{\rho}_i^A = \mathrm{tr}_B[(\mathbb{1}_A \otimes E_i)|\psi^{AB}\rangle\langle\psi^{AB}|(\mathbb{1}_A \otimes E_i)]/p_i$ and $p_i = \mathrm{tr}[(\mathbb{1}_A \otimes E_i)|\psi^{AB}\rangle\langle\psi^{AB}|(\mathbb{1}_A \otimes E_i)]$, has the following bound

$$\bar{C}_{l_1}^{A|B}(|\psi^{AB}\rangle) \leq \frac{d}{2} E(|\psi^{AB}\rangle)\bar{C}_{l_1}^{A|B}(|\Phi^{AB}\rangle), \tag{378}$$

with $|\Phi^{AB}\rangle$ being the maximally entangled state in the Schmidt decomposition basis of $|\psi^{AB}\rangle$, and for $d = 2$ case, the equality in the above equation holds. This also establishes an operational connection between created coherence of a subsystem and entanglement of the composite system, although applies only for the initial pure states.

4.5. Average coherence of randomly sampled states

As is known, some measures of quantumness manifest concentration effect, e.g., the random bipartite pure states sampled from the uniform Haar measure are typically maximally entangled (Hayden et al., 2006). Along the same line, Singh et al. (2016b) studied the coherence properties of pure states chosen randomly from the uniform Haar measure, and found that most of them possess almost the same amount of coherence which are not typically maximally coherent.

For the Haar distributed random pure states $\psi = |\psi\rangle\langle\psi|$ with dimension $d \geq 3$, they considered the average coherence of the form

$$\bar{C}(\psi) := \int d(\psi)C(\psi) = \int d\mu(U)C(U|1\rangle\langle 1|U^\dagger), \tag{379}$$

with U being sampled from the uniform Haar distribution, and $C(\psi)$ can be any faithful measure of coherence.

First, for the relative entropy of coherence, its average over all ψ was found to be

$$\bar{C}_r(\psi) = H_d - 1, \tag{380}$$

with $H_d = \sum_{k=1}^d (1/k)$ the dth harmonic number, and the logarithm in Eq. (144) is with respect to natural base here. The probability for $|C_r(\psi) - (H_d - 1)| > \epsilon$ is upper bounded by $2e^{-d\epsilon^2/36\pi^3 \ln 2 \ln^2 d}$, hence the randomly chosen ψ with $C_r(\psi)$ not close to $H_d - 1$ is exponentially small. This is the concentration phenomenon for relative entropy of coherence. It reveals that most Haar distributed random ψ have $H_d - 1$ amount of coherence, which is solely determined by the parameter d.

Second, for l_1 norm of coherence, they found that the mean classical purity $\bar{P}(\Delta[\psi])$ averaged over the Haar distributed ψ is given by $2/(d + 1)$. The probability for $|P(\Delta[\psi]) - 2/(d + 1)| > \epsilon$ is $2e^{-d\epsilon^2/18\pi^3 \ln 2}$, which is also exponentially small for $\epsilon \to 0$. Thus by using the upper bound of $C_{l_1}(\psi)$ given in Eq. (312), one has

$$\bar{C}_{l_1}(\psi) \leq \sqrt{d(d-1)^2/(d+1)}. \tag{381}$$

Finally, to show most of the Haar distributed pure states are not typically maximally coherent, Singh et al. (2016b) calculated the average trace distance between $\rho_{\mathrm{diag}}^\psi$ and the maximally mixed state ρ_{mm} (which is the optimal δ for $|\Psi_d\rangle$). The result shows that

$$\bar{\mathcal{D}}(\rho_{\mathrm{diag}}^\psi, \rho_{\mathrm{mm}}) = 2(1 - 1/d)^d, \tag{382}$$

which approaches $2/e$ in the limit of $d \to \infty$. The probability for a divergence of the amount ϵ is $2e^{-d\epsilon^2/18\pi^3 \ln 2}$, which is arbitrary small for $\epsilon \to 0$. This shows that the optimal δ for the majority of $C_r(\psi)$ are not ρ_{mm}, hence the random Haar distributed ψ are not maximally coherent.

In fact, for the uniformly distributed pure states, the average l_1 norm of coherence can be obtained. The corresponding analytical result is derived by Bu et al. (2016a), which is given by

$$\bar{C}_{l_1}(\psi) = \frac{(d-1)\pi}{4}, \tag{383}$$

and there is no concentration phenomenon for it, this is because the probability for $C_{l_1}(\psi)$ not close to $(d-1)\pi/4$ is given by $2e^{-4\epsilon^2/9d\pi^3 \ln 2}$, which is finite when $d \to \infty$. But the scaled l_1 norm of coherence $C_{l_1}(\rho)/(d-1)$ concentrates around $\pi/4$ for very large values of d, and the probability for a divergence of the amount ϵ is given by $2e^{-4(d-1)^2\epsilon^2/9d\pi^3 \ln 2}$.

Similarly, for d-dimensional randomly mixed states sampled from various induced measures, Zhang et al. (2017c) considered the average relative entropy of coherence

$$\begin{aligned}\bar{C}_r(\alpha,\gamma) &:= \int d\mu_{\alpha,\gamma}(\rho) C_r(\rho) \\ &= \int d\mu_{\alpha,\gamma}(U\Lambda U^\dagger) C_r(U\Lambda U^\dagger),\end{aligned} \tag{384}$$

with $\Lambda = \mathrm{diag}\{\lambda_1, \ldots, \lambda_d\}$, and $U\Lambda U^\dagger$ is the isospectral full-ranked density matrices (i.e., the spectra of Λ is nondegenerate), and $\mu_{\alpha,\gamma}$ is the normalized probability measure on the set of density matrix $\mathcal{D}(\mathbb{C}^d)$.

For the special case of mixed ρ sampled from induced measures obtained via partial tracing of the Haar distributed dd'-dimensional ($d' \geq d$) pure states ψ, the average coherence can be further obtained analytically as $\bar{C}_r(\alpha,\gamma) = (d-1)/2d'$ for $(\alpha,\gamma) = (d'-d+1, 1)$. If d is further restricted to $d \geq 3$, the probability for $|C_r(\alpha,\gamma) - (d-1)/2d'| > \epsilon$ is bounded from above by $2e^{-dd'\epsilon^2/144\pi^3 \ln 2 \ln^2 dd'}$. Hence, nearly all ρ obtained via partial tracing over the uniformly Haar distributed random pure bipartite states ψ in the Hilbert space $\mathcal{H}_{d'}$ have coherence approximately equal to the average relative entropy of coherence. These results were further extended by the same author in a recent work (Zhang, 2017).

5. Quantum coherence in quantum information

5.1. Quantum state merging

For a quantum protocol with two or more parties, e.g., the simplest case of two players, Alice and Bob, one may wonder how much coherence is localized (or consumed) at Bob's side, and simultaneously, how much entanglement is established (or consumed) for Alice and Bob, after finishing the pre-designed computation procedure?

Streltsov et al. (2016) explored such a problem. They discussed the protocol of quantum state merging under IO, which they called incoherent quantum state merging, and is indeed an analog of the standard state merging with general quantum operations (Horodecki et al., 2005c). In this task, Alice, Bob, and a referee share the state ρ^{RAB}. Alice and Bob also have access to Φ_2 at rate E, and Bob has access to Ψ_2 at rate C. The goal is for them to merge the state of AB on Bob's side by LQICC, i.e., Alice performs general quantum operations, while Bob is restricted to IO only.

By denoting $E = E_i - E_t$ and $C = C_i - C_t$, with E_i and E_t (C_i and C_t) being the entanglement rate of AB (local coherence of B) before and after the state merging protocol, Streltsov et al. (2016) showed that the entanglement–coherence pair (E, C) is achievable if there exists E_i, E_t, C_i, C_t, and sufficiently large integers n such that

$$\begin{aligned}&\| \Lambda_{QI}[\rho_i^{\otimes n} \otimes \Phi_2^{\lfloor (E_i+\delta)n \rfloor} \otimes \Psi_2^{\otimes \lfloor (C_i+\delta n) \rfloor}] \\ &- \rho_t^{\otimes n} \otimes \Phi_2^{\otimes \lceil E_t n \rceil} \otimes \Psi_2^{\otimes \lceil C_t n \rceil} \|_1 \leq \varepsilon,\end{aligned} \tag{385}$$

is satisfied for every $\varepsilon > 0$ and $\delta > 0$. Moreover, $\rho_i = \rho^{RAB} \otimes |0^{\tilde{B}}\rangle\langle 0^{\tilde{B}}|$, $\rho_t = \rho^{R\tilde{B}B} \otimes |0^A\rangle\langle 0^A|$, and $|0^{\tilde{B}}\rangle$ is the initial state of the auxiliary system $\tilde{B}$ (with the same dimension as A) belong to Bob. $E > 0$ ($C > 0$) corresponds to entanglement (coherence) consumption in the task of state merging, while $E < 0$ ($C < 0$) corresponds to the reverse situation, i.e., the merging protocol is achievable for free, with the additional gain of entanglement (coherence) at rate $|E|$ ($|C|$).

On the above basis, Streltsov et al. (2016) found that the sum of E and C is upper bounded by a nonnegative quantity, i.e.,

$$E + C \leq S(\Delta^{AB}[\rho^{RAB}]) - S(\Delta^B[\rho^{RAB}]), \tag{386}$$

where Δ^{AB} and Δ^B are the same as that in Eq. (364). The equality holds for any pure state ρ^{RAB}, for which (E, C) reduces to $(E_0, 0)$, with $E_0 = S(\bar{\rho}^{AB}) - S(\bar{\rho}^B)$ and $\bar{\rho}^{AB} = \Delta(\rho^{AB})$. It implies that whenever $E < 0$, we must have $C \geq 0$, and vice versa. Therefore, there is no state merging procedure for which entanglement and coherence can be gained simultaneously. This can be recognized as an operational complementarity relation between entanglement of a bipartite state and quantum coherence of its reduction.

5.2. Deutsch–Jozsa algorithm

The Deutsch–Jozsa algorithm is one of the first quantum algorithms in quantum information science. It uses quantum coherence as a resource, and this enables its speedup compared with that of the classical counterpart (Deutsch and Jozsa, 1992).

By considering a discrete quantum walk version, Hillery (2016) studied the Deutsch–Jozsa algorithm. It is performed in three steps: (i) let the particle sitting at the edge between 0 and A (with the state denoting as $|0, A\rangle$) traverses the vertex A, which transforms it to $U_A|0, A\rangle = \sum_{j=0}^{N} |A, j\rangle / \sqrt{(N+1)}$; (ii) It goes further from A to B between which there are N paths, and there will be a phase $e^{i\phi_j}$ ($\phi_j = 0$ or π) being added after traversing the vertex of the jth path, thus the state becomes

$(|0,-1\rangle+\sum_{j=1}^{N}e^{i\phi_j}|j,B\rangle)/\sqrt{(N+1)}$; (iii) Finally, the particle traversing the vertex B and it is transformed into

$$\frac{1}{\sqrt{N+1}}|-1,-2\rangle+\frac{1}{N+1}\sum_{j,k=1}^{N}e^{i\phi_j}e^{2\pi ijk/(N+1)}|B,k\rangle + \frac{1}{N+1}\sum_{j=1}^{N}e^{i\phi_j}|B,N+1\rangle. \tag{387}$$

To discuss in a quantitative way how quantum coherence affects performance of the algorithm, Hillery (2016) further introduced a qubit (with the initial state $|0\rangle$) to every path of the graph. By supposing the qubit state $|\mu_j\rangle=\alpha_j|0\rangle_j+\beta_j|1\rangle_j$ after traversing the jth vertex, and defining $|\eta\rangle_j=|\mu_j\rangle\prod_{k\neq j}^{N}|0\rangle_k$ and $|\eta\rangle_0=\prod_{k=0}^{N}|0\rangle_k$, their state after passing through the N paths will be

$$|\Psi\rangle_{in}=(|0,-1\rangle|\eta_0\rangle+\sum_{j=1}^{N}e^{i\phi_j}|j,B\rangle|\eta_j\rangle)/\sqrt{(N+1)}, \tag{388}$$

for which the l_1 norm of coherence is given by $C_{l_1}(|\Psi\rangle_{in})=\sum_{j\neq k}^{N}|\langle\eta_k|\eta_j\rangle|/(N+1)$, and the output state after the vertex B is

$$\begin{aligned}|\Psi\rangle_{out}=&\frac{1}{\sqrt{N+1}}|-1,-2\rangle|\eta_0\rangle\\&+\frac{1}{N+1}\sum_{j,k=1}^{N}e^{i\phi_j}e^{2\pi ijk/(N+1)}|B,k\rangle|\eta_j\rangle\\&+\frac{1}{N+1}\sum_{j=1}^{N}|\eta_j\rangle e^{i\phi_j}|B,N+1\rangle,\end{aligned} \tag{389}$$

then the probability of finding the particle on the edge between B and $N+1$ is

$$\begin{aligned}p&=|\langle B,N+1|\Psi\rangle_{out}|^2\\&=\frac{1}{(N+1)^2}\sum_{j,k=1}^{N}e^{i(\phi_j-\phi_k)}\langle\eta_k|\eta_j\rangle\\&\leq\frac{N}{(N+1)^2}+\frac{C_{l_1}(|\Psi\rangle_{in})}{N+1}.\end{aligned} \tag{390}$$

Clearly, the amount of coherence in the system limits our ability to distinguish between the constant case (i.e., all ϕ_j are the same and thus p takes the maximum value) and the balanced case (half of ϕ_j are zero and half of ϕ_j are π, thus p takes the minimum value).

When one have not detect the particle in the edge between B and $N+1$, one can guess we have the balanced case. Hillery (2016) calculated the error probability for the classical and quantum Deutsch–Jozsa algorithm after m trials of the discrete quantum walk experiments, and found they are given respectively, by

$$p_{error}^{\text{class}}=\frac{1}{2^m},\ p_{error}^{\text{quant}}=\frac{1}{2}(1-\upsilon)^m, \tag{391}$$

with $\upsilon=\langle\eta_k|\eta_j\rangle$ is supposed to be positive for all $j\neq k$. Thus if υ is larger than a critical value, the quantum algorithm always outperforms its classical counterparts.

5.3. Grover search algorithm

The Grover search algorithm is another important algorithm in the developments of quantum information science (Grover, 1997). The pursue of the reason for the speedup of this algorithm attract researchers' interest for many years.

For an N-qubit database initialized as

$$|\psi_0\rangle=\sqrt{\frac{j}{N}}|X\rangle+\sqrt{\frac{N-j}{N}}|X^{\perp}\rangle, \tag{392}$$

where $|X\rangle=\sum_{x_s}|x_s\rangle/\sqrt{j}$, $|X^{\perp}\rangle=\sum_{x_n}|x_n\rangle/\sqrt{N-j}$, and j represents the number of solutions. To optimize the success probability, one can perform the Grover operation

$$G=OD, \tag{393}$$

with $O = \mathbb{1} - 2|X\rangle\langle X|$ and $D = 2|\psi_0\rangle\langle\psi_0| - \mathbb{1}$. After r iterations of the Grover operation G, the initial state $|\psi_0\rangle$ turns to be

$$|\psi_r\rangle \equiv G^r|\psi_0\rangle = \sin\alpha_r|X\rangle + \cos\alpha_r|X^\perp\rangle, \tag{394}$$

with $\alpha_r = (2r+1)\arctan\sqrt{j/(N-j)}$. The success probability for finding the correct result is $p(r) = \sin^2\alpha_r$, and the optimal times of search is given by $r_{\mathrm{opt}} = CI[(\pi-\alpha)/2\alpha]$, with $CI[x]$ denoting the closest integer to x.

Shi et al. (2017) calculated the relative entropy and the l_1 norm of coherence for $|\psi_r\rangle$, and found that the success probability $p(r)$ depends on the amount of quantum coherence remaining in $|\psi_r\rangle$. To be explicit,

$$\begin{aligned} C_r(|\psi_r\rangle) &= H(p) + \log_2(N-j) + p\log_2\frac{j}{N-j}, \\ C_{l_1}(|\psi_r\rangle) &= \left[\sqrt{jp} + \sqrt{(N-j)(1-p)}\right]^2 - 1, \end{aligned} \tag{395}$$

both of which decrease with the increasing value of p. Therefore, the larger the quantum coherence depletion (or equivalently, the less the remaining quantum coherence in $|\psi_r\rangle$), the bigger the success probability one can obtain.

Shi et al. (2017) also calculated the quantum coherence depletion for the generalized Grover search algorithm (Biham et al., 1999), and found that the required optimal search time may increase with the increasing quantum coherence depletion. Moreover, quantum correlations such as quantum entanglement and QD cannot be directly related to the success probability or the optimal search time.

5.4. Deterministic quantum computation with one qubit

The DQC1 algorithm is the first algorithm that shows quantum computation can outperform those of the classical computation even without entanglement (Jozsa and Linden, 2003; Laflamme et al., 2002). The standard DQC1 algorithm starts with an initial product state $|0\rangle\langle 0| \otimes (\mathbb{1}/2^n)$, and then it was transformed into

$$\tilde{\rho}^{AR} = \frac{1}{2}\left(\mathbb{1}_2 \otimes \frac{\mathbb{1}_{2^n}}{2^n} + |0\rangle\langle 1| \otimes \frac{U^\dagger}{2^n} + |1\rangle\langle 0| \otimes \frac{U}{2^n}\right), \tag{396}$$

after performing a Hadamard operation on the first qubit, who then served as the control qubit when a controlled unitary operation U is performed on the target qubits in the maximally mixed state $\mathbb{1}_{2^n}/2^n$ (Knill and Laflamme, 1998). The goal of this algorithm is to estimate the normalized trace of U.

As the reduced states of the control qubit after the series operations is given by

$$\tilde{\rho}^A = \frac{1}{2}\begin{pmatrix} 1 & \dfrac{\mathrm{tr}U^\dagger}{2^n} \\ \dfrac{\mathrm{tr}U}{2^n} & 1 \end{pmatrix}, \tag{397}$$

the estimation can be finalized by measurements of the ancilla in an appropriate basis, i.e., $\langle\sigma_x + i\sigma_y\rangle_{\tilde{\rho}_A} = \mathrm{tr}U/2^n$.

For the above fashion of DQC1, several works have been undertaken to understand the origin of its superiority over the classical algorithm (Dakić et al., 2010; Datta and Gharibian, 2009; Datta et al., 2008). Recently, Ma et al. (2016a) provided a further viewpoint of its superiority from the perspective of quantum coherence. By choosing the computational basis for the ancilla qubit and the eigenbasis of U for the register as the reference basis, they studied how the interplay between coherence consumption and creation of QD works in DQC1, and showed that

$$\bar{D}(\tilde{\rho}^{AR}) \le \delta C(\rho^A),\ \ \bar{D}_{R|A}(\tilde{\rho}^{AR}) \le \delta C(\rho^A), \tag{398}$$

which are direct consequences of Eqs. (280) and (281). When being measured by the relative entropy, the coherence consumption can be obtained (note that ρ^A is maximally coherent) from Eq. (396) as

$$\delta C(\rho^A) = C_r(\rho^A) - C_r(\tilde{\rho}^A) = H_2\left(\frac{1-|\mathrm{tr}U|/2^n}{2}\right), \tag{399}$$

where $H_2(\cdot)$ is the binary Shannon entropy function. It shows that the speedup of this algorithm always corresponds to the consumption of quantum coherence in the ancilla. When there is no coherence to be consumed, we must have $|\mathrm{tr}U| = 2^n$, and thus $U = e^{i\phi}\mathbb{1}$ for some ϕ.

By considering a duplication of the DQC1 protocol termed as nonlocal deterministic quantum computation with two qubits (NDQC2), i.e., the collaborative task of estimating the product of normalized traces of two unitaries without obtaining the individual trace value of each unitary, Shahandeh et al. (2017) found that its computational advantage can be achieved with quantum states that have no quantum entanglement and QD. To interpret this phenomenon, they introduced an operational definition of nonclassical correlations, that is, a state ρ^{AB} is said to be nonclassical if it enables a collaborative task only using correlated inputs and measurement results of correlations more efficiently than any classical algorithm. Based on this framework, they defined

$$C^{\mathrm{net}}(\rho^{AB}) = C(\rho^{AB}) - C(\rho^A) - C(\rho^B), \tag{400}$$

and suggested that this quantity can be used for interpreting the efficiency of the NDQC2 protocol, as its quantum advantage is achieved only when $C^{\mathrm{net}}(\rho^{AB}) > 0$.

For the relative entropy of coherence, from Eq. (296) it is clear that $C_r^{\mathrm{net}}(\rho^{AB}) \geq 0$, and it takes the maximum for the maximally coherent states of the form of Eq. (140). Moreover, ρ^{AB} is a classical–classical state if and only if $C^{\mathrm{net}}(\rho^{AB}) = 0$ for certain reference bases. Operationally, when there are no local coherence, i.e., when $C(\rho^A) = C(\rho^B) = 0$, two spatially separated parties (Alice and Bob) cannot distil quantum coherence on neither sides using LICC if and only if $C_r^{\mathrm{net}}(\rho^{AB}) = 0$. This shows another physical implication of the net global coherence as a primitive property of quantum systems which is distinct from those captured by entanglement or QD.

Based on the aforementioned facts, Shahandeh et al. (2017) also gave a basis dependent characterization of nonclassical states, that is, a state ρ^{AB} is said to be nonclassical if and only if

$$C_{r,\max}^{\mathrm{net}}(\rho^{AB}) = \max_{\{|i\rangle_A|j\rangle_B\}} C_r^{\mathrm{net}}(\rho^{AB}) > 0, \tag{401}$$

and it vanishes only for the product states $\rho^{AB} = \rho^A \otimes \rho^B$. On the contrary, as $C_r^{\mathrm{net}}(\rho^{AB}) = I(\rho^{AB}) - I(\Delta[\rho^{AB}])$, then when it is minimized over the reference bases, we obtain the symmetric discord (Hu and Fan, 2017), see Eq. (284).

5.5. Quantum metrology

Considering an explicit metrology task, i.e., the phase discrimination (PD) game. In this game, a particle in the state $\rho \in \mathcal{D}(\mathbb{C}^d)$ passes through a black box, after which an unknown phase was encoded to it as $U_\phi \rho U_\phi^\dagger$, with $U_\phi = \sum_{j=0}^{d-1} e^{ij\phi}|j\rangle\langle j|$, $\phi \in \mathbb{R}$, and $\{|j\rangle\}$ being the reference basis. For a collection of pairs $\Theta = \{p_k, \phi_k\}_{k=0}^{m-1}$, the goal of the PD game is to predict the phases $\{\phi_k\}$ with success probability as high as possible. In general, the optimal probability can be obtained by optimizing over all measurements $\{M_k\}$, and it is given by

$$p_\Theta^{\mathrm{succ}}(\rho) = \max_{\{M_k\}} \sum_k p_k \mathrm{tr}[U_{\phi_k} \rho U_{\phi_k}^\dagger \rho], \tag{402}$$

Napoli et al. (2016) showed that for the above metrology task, the optimal probability can be linked to RoC of the state ρ, i.e.,

$$\max_\Theta \frac{p_\Theta^{\mathrm{succ}}(\rho)}{p_\Theta^{\mathrm{succ}}(\mathcal{I})} = 1 + C_R(\rho), \tag{403}$$

where the maximum is achieved for $\Theta^\star = \{1/d, 2\pi k/d\}_{k=0}^{d-1}$, and $p_\Theta^{\mathrm{succ}}(\mathcal{I})$ is the corresponding classical probability obtained only by guessing. Therefore, $C_R(\rho)$ quantifies the quantum advantage of the PD task, thus suggests a prominent role of RoC in quantum information processing.

Yu (2017) also investigated a similar metrology task. The difference is that they linked the skew-information-based coherence measure $C_{sk}(\rho)$ in Eq. (226) to uncertainty of the estimated phase. By using the quantum Cramér–Rao bound

$$(\delta\phi_k)^2 \geq \frac{1}{N\mathcal{F}(\rho_\phi)}, \tag{404}$$

with $\mathcal{F}(\rho_\phi)$ being the quantum Fisher information given in Eq. (133), they showed that

$$\frac{1}{4NC_{sk}(\rho)} \leq \sum_k (\delta\phi_k^\star)^2 \leq \frac{1}{8NC_{sk}(\rho)}, \tag{405}$$

where $\delta\phi_k^\star$ denotes the optimal variance. It shows that the measurement precision can be increased by increasing RoC of the state ρ.

By considering the subchannel discrimination task which is a generalization of the PD task, Bu et al. (2017b) provided an operation interpretation for the maximum relative entropy of coherence defined in Eq. (216). Here, the subchannel refers to a map that is linear completely positive and trace nonincreasing, and an instrument $\mathfrak{J}$ for a channel $\mathcal{E} = \sum_a \mathcal{E}_a$ is a collection of subchannels $\{\mathcal{E}_a\}$. The optimal probability of successfully discriminating the subchannels in $\mathfrak{J}$ reads

$$P_{\mathfrak{J}}^{\mathrm{succ}}(\rho) = \max_{\{M_k\}} \sum_a \mathrm{tr}[\mathcal{E}_a(\rho) M_k], \tag{406}$$

where the optimization is taken over the POVM $\{M_k\}$. If we are restricted to the set of incoherent states, the resulting optimal probability turns out to be $P_{\mathfrak{J}}^{\mathrm{succ}}(\mathcal{I}) = \max_{\sigma\in\mathcal{I}} P_{\mathfrak{J}}^{\mathrm{succ}}(\sigma)$. Bu et al. (2017b) showed that the optimal maximum advantage achievable in subchannel discrimination can be characterized by the maximum relative entropy of coherence. To be precise, we have

$$2^{C_{\max}(\rho)} = \max_{\mathfrak{J}} \frac{P_{\mathfrak{J}}^{\mathrm{succ}}(\rho)}{P_{\mathfrak{J}}^{\mathrm{succ}}(\mathcal{I})}, \tag{407}$$

which is very similar to Eq. (402) as $C_{\max}(\rho)$ is connected to $C_R(\rho)$ via $C_{\max}(\rho) = \log_2[1 + C_R(\rho)]$, see Eq. (217).

6. Quantum correlations and coherence under quantum channels

As a precious physical resource for implementing quantum computation and communication tasks that are otherwise impossible classically, and due to the obvious fact that nearly all quantum systems inevitably interact with their surroundings which may cause decoherence and other negative effects, the study of GQD and quantum coherence, in particular, the control and maintenance of them in noisy environments, is of equal importance to the study of other similar problems such as quantum correlation measures (Modi et al., 2012; Xu and Li, 2013).

6.1. Frozen phenomenon of QD and quantum coherence

6.1.1. Freezing of quantum discord

Hassan and Joag (2013) investigated the family of local quantum channels under the action of which the QD is preserved for all bipartite states. By using a result of Petz (2003) which says that

$$S(\rho \parallel \sigma) = S(T[\rho] \parallel T[\sigma]), \tag{408}$$

if and only if the map $\rho \mapsto T[\rho]$ and $\sigma \mapsto T[\sigma]$ are invertible, they showed that the mutual-information-based QD is frozen for all states if and only if the channels are invertible. Explicitly, by denoting Λ_A (Λ_B) the quantum channel acting on party A (B), then

$$D_A(\rho^{AB}) = D_A(\Lambda_A \otimes \Lambda_B[\rho^{AB}]), \tag{409}$$

if and only if there exists Λ_A^* and Λ_B^* such that

$$\begin{aligned} &(\Lambda_A^* \otimes \Lambda_B^*)(\Lambda_A \otimes \Lambda_B)[\rho^{AB}] = \rho^{AB}, \\ &(\Lambda_A^* \otimes \Lambda_B^*)(\Lambda_A \otimes \Lambda_B)[\rho^{A} \otimes \rho^{B}] = \rho^{A} \otimes \rho^{B}. \end{aligned} \tag{410}$$

Moreover, for a distance measure of two states that is monotonic under the action of quantum channel, the corresponding GQD defined based on it is frozen if and only if the local quantum channels Λ_A and Λ_B are invertible. The related distance measures include those based on the trace norm and Uhlmann fidelity.

For certain quantum channels, the QD may be frozen for a restricted family of states. You and Cen (2012) studied such a problem. They considered the phase damping channel whose action on a state can be described by $\Lambda_{pd}(\rho) = \sum_i E_i \rho E_i^\dagger$, with

$$E_0 = \mathrm{diag}\{1, p(t)\},\ E_1 = \mathrm{diag}\{0, \sqrt{1 - p^2(t)}\}, \tag{411}$$

being the Kraus operators, and $p(t)$ a time-dependent parameter containing the information of the channel. For the initial Bell-diagonal states ρ^{Bell} of Eq. (34) with one subsystem subjecting to the channel Λ_{pd}, they obtained necessary and sufficient conditions for freezing QD, which are given in terms of the triple (c_1, c_2, c_3). Explicitly, the QD in ρ^{Bell} is frozen if and only if

$$\begin{aligned} &c_2 = -c_1 c_3,\ |c_1| > |c_3| \\ \text{or } &c_1 = -c_2 c_3,\ |c_2| > |c_3|. \end{aligned} \tag{412}$$

The above condition is also of special importance for studying the universal freezing of geometric quantum correlations (Cianciaruso et al., 2015). Besides these, they also generalized their results to an extended family of two-qubit states

$$\rho = \rho^{\mathrm{Bell}} + \frac{1}{4}(c_{12}\sigma_1 \otimes \sigma_2 + c_{21}\sigma_2 \otimes \sigma_1), \tag{413}$$

and obtained a similar necessary and sufficient condition.

Haikka et al. (2013) also studied the frozen phenomenon of QD in dephasing reservoir. The difference is that they considered the explicit Ohmic-type spectrum given by

$$J(\omega) = \omega^s \omega_c^{1-s} e^{-\omega/\omega_c}, \tag{414}$$

with ω_c being cutoff frequency of the reservoir, and it is said to be sub-Ohmic if $0 < s < 1$, Ohmic if $s = 1$, and super-Ohmic if $s > 1$. For a subset of the initial Bell-diagonal state

$$\rho_{\mathrm{sub}}^{\mathrm{Bell}} = \frac{1+c}{2}|\Psi^\pm\rangle\langle\Psi^\pm| + \frac{1-c}{2}|\Phi^\pm\rangle\langle\Phi^\pm|, \tag{415}$$

with $|\Psi^\pm\rangle = (|00\rangle \pm |11\rangle)/\sqrt{2}$ and $|\Phi^\pm\rangle = (|01\rangle \pm |10\rangle)/\sqrt{2}$, they obtained the expression of the evolved QD, and found that if $e^{-\Lambda(\bar{t})} = c$ [$\Lambda(t)$ is the dephasing factor], then there will be a transition from classical decoherence to quantum decoherence. But if there is no solution for $e^{-\Lambda(\bar{t})} = c$, the QD will be frozen forever, with the time-invariant value

$$D_A(\rho_{\mathrm{sub}}^{\mathrm{Bell}}) = \frac{1+c}{2}\log_2(1+c) + \frac{1-c}{2}\log_2(1-c). \tag{416}$$

The (s, c) region for which the frozen condition is satisfied is determined by temperature of the reservoir. For the zero-temperature case, they obtained numerically the corresponding (s, c) region, which shrinks with the increase of c and vanishes when $c \gtrsim 0.16$.

For two qubits prepared initially in the Bell-diagonal states described by the triple (c_1, c_2, c_3), there may be universal freezing of GQD defined based on the distance measures $\mathcal{D}$ of quantum states that satisfy the following conditions: (i) contractivity under CPTP maps, (ii) invariant under transposition, and (iii) convexity under mixing of states. Distance of this type include the relative entropy, the squared Bures distance, the squared Hellinger distance, and the trace distance.

From the above conditions, Cianciaruso et al. (2015) considered the initial Bell-diagonal state ρ^{Bell} of Eq. (34) with the triple $(c_1, -c_1c_3, c_3)$, and proved that its distance to $(c_1, 0, 0)$ $[(0, 0, c_3)]$ is independent of c_1 (c_3). Moreover, one of the closest classical state to ρ^{Bell} is still a Bell-diagonal state (s_1, s_2, s_3) with however only one of s_k is nonzero, and for the special case $c_2 = -c_1c_3$, the closest classical state further reduces to $(c_1, 0, 0)$ if $|c_1| \geqslant |c_3|$, and $(0, 0, c_3)$ otherwise. This extends the results of Eq. (412). As it shows Eq. (412) holds for general distance measure of states satisfying the above three conditions.

Based on these formulas, Cianciaruso et al. (2015) found that when the two qubits are subject to independent phase flip (similar for bit flip and bit-phase flip) channels, the GQDs satisfying the above three conditions will be frozen in the time interval $t < t^* = -(1/2\gamma)\ln(|c_3(0)|/|c_1(0)|)$. As this conclusion depends only on the proposed properties of distance measures of states, it shows the universal freezing of geometric quantum correlations.

Montealegre et al. (2013) studied the trace norm of discord for a two-qubit system (initially prepared in the Bell-diagonal state) passes through the local bit flip, phase flip, bit-phase flip, and generalized amplitude damping channels. Through detailed analysis with different initial state parameters (c_1, c_2, c_3), they found that the trace norm of discord exhibits the phenomenon of freezing behavior during its evolution process. Aaronson et al. (2013b) discussed the trace norm of discord with the locally applied phase-flip channels and random external fields and observed the freezing phenomenon. They also compared dynamics of the total and classical-classical correlations defined via the trace norm, see Eq. (43). Moreover, the trace norm, Bures distance, and Hellinger distance measure of GQD for two non-interacting qubits subject to two-sided and one-sided thermal reservoirs have also been investigated (Hu and Sun, 2015). In fact, the frozen phenomenon of various GQDs were proved to be universal by Aaronson et al. (2013a) and Cianciaruso et al. (2015).

6.1.2. *Freezing of quantum coherence*

For a N-qubit quantum system subject to local independent and identical decohering environments, Bromley et al. (2015) studied decay dynamics of coherence and provided important insights between them and the discordlike correlation measures. The extension of two-qubit Bell-diagonal states, i.e.,

$$\rho_N^{\text{Bell}} = \frac{1}{2^N}\left(\mathbb{1}_2^{\otimes N} + \sum_{i=1}^{3} c_i\sigma_i^{\otimes N}\right), \tag{417}$$

are also described by the triple (c_1, c_2, c_3). For a system with even number of qubits, Bromley et al. (2015) found that if

$$c_2 = (-1)^{N/2}c_1c_3, \tag{418}$$

then all bona fide distance-based coherence measures will be permanently frozen for local bit flip channel (similar result can be obtained for local bit-phase flip channel by exchanging c_1 and c_2). These include the relative entropy of coherence for general even N, and the trace norm of coherence for $N = 2$.

Moreover, for general one-qubit state (i.e., ρ_N^{Bell} with $N = 1$) subject to bit flip channel, the l_1 norm of coherence is frozen forever if $c_2 = 0$, while for the two-qubit state of Eq. (22) with the elements of T vanishing for all non-diagonal elements, it is frozen when the parameters

$$x_2 = y_2 = 0,\ T_{22} = uT_{11},\ \ (-1 \le u \le 1). \tag{419}$$

Experimentally, the freezing phenomenon for relative entropy of coherence (Baumgratz et al., 2014), fidelity-based measure of coherence (Streltsov et al., 2015), and trace norm of coherence (Bromley et al., 2015) for two and four qubits exposing to the phase damping channel were observed in a nuclear magnetic resonance system (Silva et al., 2016).

If an incoherent operation satisfies not only $K_i\mathcal{I}K_i^\dagger \subset \mathcal{I}$, but also the additional constraint $K_i^\dagger\mathcal{I}K_i \subset \mathcal{I}$ for all K_i, then it is said to be strictly incoherent (Winter and Yang, 2016). Their Kraus operators contain at most one nonzero entry in each row and each column, and incoherent channels of such type cover the paradigmatic source of noises in quantum information science, e.g., the bit flip, phase flip, bit-phase flip, depolarizing, amplitude damping, and phase damping channels.

By restricting to strictly incoherent channels, Yu et al. (2016b) established a measure-independent freezing condition of coherence, which states that for any initial state of a system, all measures of its coherence are frozen if and only if its relative entropy of coherence is frozen. The proof for this claim comprises two essential steps. First, if $\delta^\star$ is the closest incoherent state to ρ in the definition of $C_r(\rho)$, then $\Lambda[\delta^\star]$ is the closest state to $\Lambda[\rho]$ for $C_r(\Lambda[\rho])$. Second, if the channel maps $\rho(0)$ to $\rho(t)$, i.e., $\Lambda[\rho(0)] = \rho(t)$, then one can always construct an incoherent operation which gives the map $\mathcal{R}[\rho(t)] = \rho(0)$.

For a system of N qubits interacting independently with N bit flip (not necessary to be identical) channels, Yu et al. (2016b) further identified two families of states for which all measures of quantum coherence are frozen, they are given respectively by:

(1) $|\varphi_l^{\pm}\rangle = (|l\rangle \pm |\bar{l}\rangle)/\sqrt{2}$, with the sequences $l = l_1 l_2 \cdots l_N$, $\bar{l} = \bar{l}_1\bar{l}_2\cdots\bar{l}_N$, $l_1 = 0$, $l_{i\neq 1} = \{0,1\}$, $\bar{l}_i = 1 - l_i$.
(2) $\rho = \sum_l p_l [p|\varphi_l^+\rangle\langle\varphi_l^+| + (1-p)|\varphi_l^-\rangle\langle\varphi_l^-|]$, with $0 \le p \le 1$, and p_l being any probability distribution.
By decomposing state ρ as Eq. (154), and using a transformation matrix T to describe the action of $\mathcal{E}$, i.e.,

$$\mathcal{E}^{\dagger}(X_i) = \sum_j T_{ij} X_j, \tag{420}$$

Hu and Fan (2016a) also derived a condition for freezing the l_1 norm of coherence. They found that when $T_{k0} = 0$ for $k \in \{1, 2, \ldots, d^2 - d\}$, and T^S (the submatrix of T consisting T_{ij} with i ranging from 1 to $d^2 - d$ and j from 1 to $d^2 - 1$) is a rectangular block diagonal matrix, with the main diagonal blocks

$$T_r^S = \begin{pmatrix} T_{2r-1,2r-1} & T_{2r-1,2r} \\ T_{2r,2r-1} & T_{2r,2r} \end{pmatrix} \ (r \in \{1, \ldots, d_0\}), \tag{421}$$

being orthogonal matrices, i.e., $(T_r^S)^T T_r^S = \mathbb{1}_2$, the l_1 norm of coherence for $\rho^{\hat{n}}$ will be frozen during the entire evolution. Here, $\rho^{\hat{n}}$ represents states with the characteristic vectors $\vec{x}$ [see Eq. (154)] along the same or completely opposite directions but possessing different lengths.

6.2. Enhancing the quantum resources via quantum operations

Since QD and quantum coherence are both quantum properties of quantum states, the ability of a quantum channel to create and/or enhance strength of QD or quantum coherence is related to the quantumness of the channel. It is then of interest to study whether a channel has the ability to create quantum resources, and how many quantum resources the channel can create or enhance.

6.2.1. Creation of quantum discord from classical states

When a bipartite system is coupled to a common bath, it was proved that a Markovian dissipative quantum channel can generate QD from some bipartite product states if and only if it cannot be reduced to individual decoherence channels independently acting on each qudit (Hu et al., 2011). Further, if the subsystems initially share classical correlations, even local operations can create QD.

The local creation of quantum correlations was first studied by Streltsov et al. (2011a). They investigated the completely decohering (or semiclassical) channel Λ_{sc} described by

$$\Lambda_{sc}(\rho) = \sum_k p_k(\rho)|k\rangle\langle k|, \tag{422}$$

and the unital channel Λ_u which keeps the maximally mixed state invariant. For a single qubit of a multiqubit system subject to a channel, they proved strictly that Λ_{sc} and Λ_u are the only two types of channels that cannot create quantum correlations. Equivalently, for qubit systems, a necessary and sufficient condition for a local channel to create quantum correlation is that the channel is neither completely decohering nor unital. The geometric quantum correlations defined based on contractive distance measures of states are further showed to be nonincreasing under local semiclassical channels and local unital channels. However, this result does not hold for states with higher dimensions. For multi-qudit states with dimension $d \ge 3$, even unital channels can create quantum correlation.

Hu et al. (2012) defined commutativity-preserving channels Λ_{cp} as those preserve the commutativity of any two input states, that is,

$$[\rho, \sigma] = 0 \Rightarrow [\Lambda_{cp}(\rho), \Lambda_{cp}(\sigma)] = 0. \tag{423}$$

For any finite-dimensional multipartite systems, they proved that a channel Λ acting locally on B of a quantum–classical state can create quantum correlation if and only if $\Lambda \notin \Lambda_{cp}$. For qubit case, a commutativity-preserving channel is either a completely decohering channel or a unital channel. For qutrit case, a commutativity-preserving channel is either a completely decohering channel or an isotropic channel, which is defined as

$$\Lambda_{iso}(\rho) = p\Gamma(\rho) + (1-p)\frac{\mathbb{1}}{d}, \tag{424}$$

with p being the parameter for ensuring CPTP of Λ_{iso}, and Γ is either a unitary operation $[-1/(d-1) \leqslant p \leqslant 1]$ or is unitarily equivalent to transpose $[-1/(d-1) \leqslant p \leqslant 1/(d+1)]$.

Hu et al. (2012) also conjectured that for systems with dimension higher than 3, a commutativity-preserving channel is also either isotropic or completely decohering. Guo and Hou (2013a) further gave an affirmative answer to this conjecture. They proved that Λ acting on party B of a system cannot create QD (i.e., $D_B(\rho_{AB}) = 0 \Rightarrow D_B(\mathbb{1} \otimes \Lambda[\rho_{AB}]) = 0$) if and only if it is either a completely decohering channel or a nontrivial isotropic channel (Λ_{iso} with $p \neq 0$). Channels of these types are also showed to preserve commutativity and normality of quantum states. Furthermore, Λ_B which yields $D_B(\rho_{AB}) = 0 \Leftrightarrow D_B(\Lambda_B[\rho_{AB}]) = 0$ are restricted only to the nontrivial isotropic channels.

Ciccarello and Giovannetti (2012) considered the local creation of QD by a Markovian amplitude-damping channel described by $\Lambda_{ad}(\rho) = \sum_i E_i \rho E_i^\dagger$, with

$$E_0 = |0\rangle\langle 0| + \sqrt{1-p(t)}|1\rangle\langle 1|,\ E_1 = \sqrt{p(t)}|0\rangle\langle 1|. \tag{425}$$

For the initial state

$$\rho = \frac{1}{2}(|0\rangle\langle 0| \otimes \tau_0 + |1\rangle\langle 1| \otimes \tau_1), \tag{426}$$

with the length of local Bloch vectors for τ_0 and τ_1 equal to each other (i.e., $|\vec{s}_0| = |\vec{s}_1| = s$), they obtained

$$\begin{aligned} D_B(\rho) =& h\left(\frac{1+s|\cos(\varphi/2)|}{2}\right) + h\left(\frac{1+s|\sin(\varphi/2)|}{2}\right) \\ &- h\left(\frac{1+s}{2}\right) - 1, \end{aligned} \tag{427}$$

where φ is the angle between $\vec{s}_0$ and $\vec{s}_1$. Using this formula, they showed rigorously that Λ_{ad} acting on B can create QD from ρ. In fact, it is easy to check that Λ_{ad} is neither a completely decohering nor a unital channel, hence the local creation of QD in the present case is easy to understand from the result of Hu et al. (2012).

Now we have reviewed the conditions on the quantum channels which has the ability to increase quantum correlations. An equally important problem is to characterize the quantum states whose quantum correlations can be increased locally. Hu and Fan (2015c) studied this problem by employing the tool of quantum steering ellipsoids (Jevtic et al., 2014). They considered the amplitude damping channel acting on qubit B of a Bell-diagonal state of Eq. (34). For such a state, both $\mathcal{E}_A$ and $\mathcal{E}_B$ are unit spheres shrunk by c_1, c_2 and c_3 in the x, y and z direction, respectively. It is observed that, the local increase of discord occurs when $|c_1| \gg |c_2|, |c_3|$. An interesting consequence is that, the local quantum operation can increase the QD of an entangled state.

6.2.2. Enhancing the coherence via quantum operations

Quantum coherence measures should be monotonically decreasing under IO, which are a strict subset of non-coherence-generating (NC) channels. The behavior of different measures of coherence under the action of NC channels was studied by Hu (2016). While the relative entropy of coherence was proved monotone under all NC channels, the coherence of formation C_f can be increased by some NC channels. An example was presented that C_f of a two-qubit state is increased when a NC qubit channel is acting on one of the two qubits. Here, the NC channel is chosen as $\Lambda(\cdot) = E_1(\cdot)E_1^\dagger + E_2(\cdot)E_2^\dagger$, with

$$E_1 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ -1 & \sqrt{2} \end{pmatrix},\ E_2 = \frac{1}{2}\begin{pmatrix} 1 & \sqrt{2} \\ 1 & 0 \end{pmatrix}, \tag{428}$$

and the two-qubit input state is $\Psi^+ \equiv |\Psi^+\rangle\langle\Psi^+|$ with $|\Psi^+\rangle = (|00\rangle + |11\rangle)/\sqrt{2}$. It was checked that the C_f of the output state is strictly larger than the input state, i.e., $C_f(\mathbb{1} \otimes \Lambda(\Psi^+)) > 1 = C_f(\Psi^+)$. Interestingly, the channel $\Lambda(\cdot)$ can never increase C_f of a single-qubit state, so the ability of this channel to increase coherence is enhanced when extending to composed Hilbert space. The reason for this enhancement is that, the local NC operation turns the quantum correlation into the local coherence, and meanwhile increase the quantum coherence of the total state.

Although the amount of coherence for a state cannot be enhanced under IO by definition, this does not prevent us from obtaining probabilistically a postmeasurement state with enhanced coherence when selective measurements are allowed, e.g., by retaining those $\rho_n = K_n \rho K_n^\dagger / p_n$ [$p_n = \mathrm{tr}(K_n \rho K_n^\dagger)$] that satisfy $C(\rho_n) > C(\rho)$ and discarding the other ρ_n, one can obtain a mixed state $\sum_{n|C(\rho_n)>C(\rho)} p_n \rho_n$ with enhanced coherence. Liu et al. (2017a) considered one such problem. By taking the l_1 norm of coherence as a measure and considering the stochastic strictly incoherent operation Λ_s whose Kraus operators $\{K_n\}_{n=1}^L$ (a subset of SIO) fulfilling $\sum_{n=1}^L K_n^\dagger K_n \le \mathbb{1}$, they obtained the maximum attainable coherence for the postmeasurement state $\Lambda_s[\rho]$, which reads

$$\max_{\Lambda_s} C_{l_1}(\Lambda_s[\rho]) = \lambda_{\max}\left(\rho_{\mathrm{diag}}^{-1/2} |\rho| \rho_{\mathrm{diag}}^{-1/2}\right) - 1, \tag{429}$$

where $\lambda_{\max}(\cdot)$ is the largest eigenvalue of $\rho_{\mathrm{diag}}^{-1/2}|\rho|\rho_{\mathrm{diag}}^{-1/2}$, and $|\rho|$ is a matrix obtained from ρ by taking absolute values to all its elements. Liu et al. (2017a) also constructed the Kraus operator and the corresponding optimal probability for obtaining Eq. (429). If ρ is irreducible, by denoting $|\varphi_{\max}\rangle = (\varphi_1, \varphi_2, \ldots, \varphi_d)^T$ ($d = \dim \rho$) the eigenvector corresponding to the largest eigenvalue of $\rho_{\mathrm{diag}}^{-1/2}|\rho|\rho_{\mathrm{diag}}^{-1/2}$ and U_{in} an arbitrary incoherent unitary matrix, one has

$$\begin{aligned} K' &= \min_i \frac{\sqrt{\rho_{ii}}}{\varphi_i} U_{\mathrm{in}} \mathrm{diag}\left(\frac{\varphi_1}{\sqrt{\rho_{11}}}, \frac{\varphi_2}{\sqrt{\rho_{22}}}, \ldots, \frac{\varphi_d}{\sqrt{\rho_{dd}}}\right), \\ p_{\max}(\rho) &= \min_i \frac{\rho_{ii}}{\varphi_i^2}, \end{aligned} \tag{430}$$

and if ρ is reducible, i.e., it can be transformed by a permutation matrix into $p_1\rho_1 \oplus p_2\rho_2 \oplus \ldots \oplus p_n\rho_n \oplus \mathbf{0}$, one has

$$K' = U_{\text{in}}(K_1' \oplus K_2' \oplus \cdots \oplus K_n' \oplus \mathbf{0}),$$
$$p_{\max}(\rho) = \sum_{\lambda_{\max}^{\alpha}=\lambda_{\max}} p_\alpha p_{\max}(\rho_\alpha), \tag{431}$$

with

$$K_\alpha' = \min_i \frac{\sqrt{\rho_{ii}^\alpha}}{\varphi_i^\alpha} \operatorname{diag}\left(\frac{\varphi_1^\alpha}{\sqrt{\rho_{11}^\alpha}}, \frac{\varphi_2^\alpha}{\sqrt{\rho_{22}^\alpha}}, \ldots, \frac{\varphi_d^\alpha}{\sqrt{\rho_{dd}^\alpha}}\right). \tag{432}$$

6.2.3. Energy cost of creating quantum coherence

As it has been shown in the above sections, quantum operations acting on an incoherent state can map it to be incoherent. The amount of quantum coherence in a non-maximally coherent state can also be enhanced via some quantum operations. When the quantum operations are restricted to be unitary, it has been shown by Hu and Fan (2017), Yao et al. (2016b), and Yu et al. (2016a) that the maximal achieved relative entropy of coherence is $\log_2 d - S(\rho)$ for any initial state ρ, see Eq. (146). When ρ is incoherent, this is also the maximal coherence created by unitary operation.

In a similar manner, Misra et al. (2016) also considered the maximal creation of quantum coherence. They considered the initial state to be the thermal state of a system, and adopted eigenbasis $\{|j\rangle\}$ of the system Hamiltonian $\hat{H}$ as the reference basis. The initial thermal state $\rho^T = e^{-\hat{H}/T}$ (T is the temperature) before acting the unitary operation is incoherent. The maximum amount of relative entropy of coherence created by using unitary operations thus has the same form as Eq. (146), i.e., $C_r^{\max}(\rho^f) = \log_2 d - S(\rho^T)$, where $\rho^f = U\rho^T U^\dagger$ denotes the output state after performing the unitary operation.

To construct the corresponding optimal unitary operations, Misra et al. (2016) used the maximally coherent basis $\{|\phi_j\rangle\}$ which are very similar to that of the maximally entanglement basis (e.g., for the two-qubit case they are the four Bell states). Here, all $\{|\phi_j\rangle\}$ have maximal value of coherence, and they are orthogonal to each other. To be explicit, they can be written as

$$|\phi_j\rangle = \frac{1}{\sqrt{d}} \sum_{m=0}^{d-1} e^{i\frac{2\pi jm}{d}} |m\rangle, \tag{433}$$

which is in fact a map of $\mathbb{Z} = \sum_m e^{2\pi i m/d} |m\rangle\langle m|$ on the maximally coherent state $|\Psi_d\rangle$, and i in the superscript is the imaginary unit. Then they gave a unitary operation

$$U = \sum_{j=0}^{d-1} |\phi_j\rangle\langle j|, \tag{434}$$

for which the output state after the action of it is given by

$$\rho^f = \frac{1}{Z} \sum_{j=0}^{d-1} e^{-E_j/T} |\phi_j\rangle\langle\phi_j|, \tag{435}$$

thus $S(\rho^f_{\text{diag}}) = \log_2 d$.

On the other hand, during these processes of coherence creation and coherence enhancement, a supply of external energy is needed. Hence, it is natural to inquire if there are quantitative connections between the created quantum coherence and the amount of energy cost? In general, the energy cost is given by $\Delta E = \mathrm{tr}(U\rho U^\dagger \hat{H}) - \mathrm{tr}(\rho \hat{H})$.

For the case of initial thermal state ρ^T, by using the fact that $\mathrm{tr}(\rho^f \hat{H}) = \mathrm{tr}(\rho^f_{\text{diag}} \hat{H})$ as $\hat{H}$ is diagonal, and the maximum entropy principle which says that the thermal state has maximum entropy among all states with a fixed average energy (Jaynes, 1957a,b), one can show that with limited energy cost ΔE, the maximum created relative entropy of coherence is bounded from above by

$$C_r^{\max}(\Delta E) \le S(\rho^{T'}) - S(\rho^T), \tag{436}$$

where $\rho^{T'}$ is the thermal state at the higher temperature T' such that

$$\Delta E = \mathrm{tr}(\rho^{T'} \hat{H}) - \mathrm{tr}(\rho^T \hat{T}). \tag{437}$$

To obtain the maximal coherence with limited energy ΔE, i.e., to saturate the upper bound of Eq. (436), one should find an optimal U such that the diagonal part of ρ^f equals $\rho^{T'}$. Misra et al. (2016) proved strictly that there always exists such a (real) unitary. The derivation of such a unitary for single-qubit state is easy, but for higher dimensional case it turns out to be very complicated.

For multipartite system, Misra et al. (2016) also compared the amounts of quantum coherence and quantum total correlations (measured by the quantum mutual information) by using the same unitary operations. For the noninteracting system described by the Hamiltonian $\hat{H}=\sum_{k=1}^{N}\hat{H}_k$ ($\hat{H}_k$ is the Hamiltonian for subsystem k), and starting from the initial product thermal states $\prod_k \otimes \rho_k^T$ of each subsystems, the maximum created quantum mutual information with limited energy ΔE is given by (Huber et al., 2015)

$$I^{\max}(\Delta E)=\sum_k [S(\rho_k^{T'})-S(\rho_k^{T})], \tag{438}$$

where the optimal unitary transforms the initial state to a final state ρ^f whose all marginals are thermal states at a higher temperature T', i.e., $\rho_k^{T'}=e^{-\hat{H}_k/T'}/\mathrm{tr}e^{-\hat{H}_k/T'}$.

As ρ_{diag}^f and the products of the marginals $\prod_k \otimes \rho_k^{T'}$ have the same average energy, the maximum entropy principle implies that $S(\rho_{\text{diag}}^f)\le S(\prod_k \otimes \rho_k^{T'})$. Hence, when the maximum correlation is created among the multipartite system, the corresponding coherence is upper bounded by it. Contrary, if maximum coherence is created in the multipartite system, then the diagonal part of the output ρ^f will be a thermal state at temperature T', and it has the same average energy with the products of the marginals $\prod_k \otimes \rho_k^f$ (with $\rho_k^f = \mathrm{tr}_{l\neq k}\rho^f$) due to the product structure of $\hat{H}$, the maximum entropy principle implies $S(\rho_{\text{diag}}^f)\ge \sum_k S(\rho_k^f)$. Hence, in this case the created correlation turns to be bounded from above by the maximum created coherence. As for the problem of whether the maximum quantum coherence and correlation can be created simultaneously, the study of the two-qubit case shows that the answer to this may be negative (Misra et al., 2016).

6.3. Resource creating and breaking power

When a quantum channel has the ability to create or enhance quantum resources, it is of interest to quantify this ability. This quantification can be regarded an intrinsic property of the channel. As a dual problem, the power of a channel to decrease or destroy the quantum resources, also attracts some research interest.

6.3.1. Quantum correlating power

The quantum correlating power (QCP) is defined as the maximum amount of quantum correlations that can be created when the channel acts locally on one party of a multipartite system (Hu et al., 2013a), i.e.,

$$\mathcal{Q}(\Lambda):=\max_{\rho\in\mathcal{CQ}} Q(\Lambda\otimes \mathbb{1}(\rho)), \tag{439}$$

where Q is a bona fide measure of quantum correlation, and $\mathcal{CQ}$ denotes the set of classical–quantum states.

The QCP is an intrinsic attribute of a channel, which quantifies the channels' ability to create quantum correlations. In its definition, the maximization is taken over the set of quantum–classical states. The input states that correspond to the maximization are called the optimal input states, which are proved to be in the set of classical–classical (CC) states

$$\mathcal{CC}=\left\{\rho|\rho=\sum_i p_i|\psi_i\rangle_A\langle\psi_i|\otimes|\phi_i\rangle_B\langle\phi_i|\right\}. \tag{440}$$

where $\{|\psi_i\rangle_A\}$ and $\{|\phi_i\rangle_B\}$ are orthogonal basis of $\mathcal{H}_A$ and $\mathcal{H}_B$, respectively. The proof can be easily sketched. For any output state ρ' that corresponds to a general QC input state, one can find a CC state whose corresponding output state ρ can be transformed to ρ' by a local channel on B, i.e., $\rho'=\mathbb{1}\otimes\lambda_B(\rho)$. From the contractivity of the measure Q under CPTP map, we have $Q(\rho)\ge Q(\rho')$. Hence the definition of QCP can be optimized to

$$\mathcal{Q}(\Lambda):=\max_{\rho\in\mathcal{CC}} Q(\Lambda\otimes \mathbb{1}(\rho)). \tag{441}$$

A channel with larger amount of QCP is more quantum, in the sense of the ability to create quantum correlations. Hence it is of interest to find out the channels with the most QCP. It can be proved that, the local single-qubit channel with maximum QCP can be found in the set of the following channels

$$\mathcal{MP}=\left\{\Lambda|\Lambda(\cdot)=\sum_{i=0}^{1}|\phi_i\rangle\langle\alpha_i|(\cdot)|\alpha_i\rangle\langle\phi_i|\right\}, \tag{442}$$

where $|\phi_0\rangle$ and $|\phi_1\rangle$ are two nonorthogonal pure states.

Hu et al. (2013b) also studied the superadditivity of QCP. Its says that two zero-QCP channels can constitute a positive-QCP channel. The phase damping channel Λ_{pd} was used as an example to show how this property works. The corresponding Kraus operators are given in Eq. (411). Λ_{pd} is unital and thus $\mathcal{Q}(\Lambda_{pd})=0$. For a four-qubit initial state shared between Alice (AA') and Bob (BB')

$$\rho_{AA'BB'}=\frac{1}{4}\sum_{i,j}|ij\rangle_{AA'}\langle ij|\otimes|\psi_{ij}\rangle_{BB'}\langle\psi_{ij}|, \tag{443}$$

where

$$
\begin{aligned}
&|\psi_{00}\rangle=\frac{1}{\sqrt{2}}(|00\rangle+|11\rangle),\ |\psi_{11}\rangle=\frac{1}{\sqrt{2}}(|0+\rangle+|1-\rangle),\\
&|\psi_{01}\rangle=\frac{1}{\sqrt{2}}(|01\rangle-|10\rangle),\ |\psi_{10}\rangle=\frac{1}{\sqrt{2}}(|0-\rangle-|1+\rangle).
\end{aligned}
\tag{444}
$$

Since $|\psi_{ij}\rangle$ are orthogonal to each other, the quantum correlation on Bob is zero. After the action of $\Lambda_{pd}^{B}\otimes\Lambda_{pd}^{B'}$ on B and B', the output state becomes $\rho'_{AA'BB'}=\mathbb{1}_{AA'}\otimes\Lambda_{pd}^{B}\otimes\Lambda_{pd}^{B'}(\rho_{AA'BB'})$. Because $[\Lambda_{pd}\otimes\Lambda_{pd}(\psi_{00}),\Lambda_{pd}\otimes\Lambda_{pd}(\psi_{11})]=\frac{1}{8}\tilde{i}p\sqrt{1-p}(\sigma^{y}\otimes\sigma^{z}+\sigma_{z}\otimes\sigma^{y})\neq 0$, the output state $\rho'_{AA'BB'}$ is not a QC state. Therefore, the quantum correlation on Bob's qubits BB' is created by the channel $\Lambda_{pd}^{B}\otimes\Lambda_{pd}^{B'}$.

6.3.2. Cohering and decohering power

For a system traversing a quantum channel $\mathcal{E}$, the amount of coherence contained in it may be increased or decreased. Building upon this fact and in the same spirit as defining entangling power and discording power [see Galve et al. (2013) and references therein], one can also consider the ability of $\mathcal{E}$ on producing or destroying coherence, and introduce the concepts of cohering and decohering power of $\mathcal{E}$.

Mani and Karimipour (2015) defined the cohering power of $\mathcal{E}$ as the maximum coherence (measured in some way) that it can produce from the full set $\mathcal{I}$ of incoherent states, and the decohering power as the maximum amount of coherence lost after all the maximal-coherence-value states $\rho^{\mathrm{mcs}}\in\mathcal{M}$ (Peng et al., 2016) passing through this channel. To be precise,

$$
\begin{aligned}
&\mathrm{CP}(\mathcal{E})=\max_{\delta\in\mathcal{I}}C(\mathcal{E}[\delta]),\\
&\mathrm{DP}(\mathcal{E})=C(\rho^{\mathrm{mcs}})-\min_{\rho^{\mathrm{mcv}}\in\mathcal{M}}C(\mathcal{E}[\rho^{\mathrm{mcs}}]),
\end{aligned}
\tag{445}
$$

and the optimization can in fact be restricted to pure states, in particular, for $\mathrm{CP}(\mathcal{E})$ the maximization can be taken with only the basis states $\{|k\rangle\}$.

Focusing on single-qubit states and coherence measured by the l_1 norm and skew information, they calculated $\mathrm{CP}(\mathcal{E})$ and $\mathrm{DP}(\mathcal{E})$ for the depolarizing and bit flip (similarly for bit-phase and phase flip) channels, and showed that the unitary channel has equal cohering and decohering power in any basis, i.e., $\mathrm{CP}(U)=\mathrm{DP}(U)$. Moreover, for N qubits subjecting to N independent unitary channels, the cohering power

$$
\mathrm{CP}(\otimes_{i=1}^{N}U_i)=\prod_{i=1}^{N}[\mathrm{CP}(U_i)+1]-1,
\tag{446}
$$

while the decohering power is bounded from below by

$$
\mathrm{DP}(\otimes_{i=1}^{N}U_i)\geq 2^{N}-\prod_{i=1}^{N}[2-\mathrm{DP}(U_i)],
\tag{447}
$$

and apart from the very special case of $\mathrm{DP}(U_i)=0,\forall U_i$, $\mathrm{DP}(\otimes_{i=1}^{N}U_i)$ approaches 2^N-1 when $N\to\infty$, hence the coherence in this state will be completely deteriorated for infinitely large N.

Bu et al. (2017a) obtained the analytical solutions of the cohering power. First, when the coherence is measured by the l_1 norm, they showed that

$$
\mathrm{CP}_{l_1}(U)=\|U\|_{1\to 1}^{2}-1,
\tag{448}
$$

where $\|U\|_{1\to 1}=\max_{1\leq j\leq d}\{\sum_{i=1}^{d}|U_{ij}|\}$ is the matrix norm, and for N-qubit system, the Hadamard gate $H^{\otimes N}$ [with $H=(\sigma_x+\sigma_z)/\sqrt{2}$] was shown to have maximum cohering power. When we adopt the relative entropy of coherence, it is given by

$$
\mathrm{CP}_{r}(U)=\max_{1\leq j\leq d}H(|U_{1j}|^2,\ldots,|U_{dj}|^2),
\tag{449}
$$

where $H(p_1,\ldots,p_d)$ denotes the Shannon entropy.

For general quantum channel $\mathcal{E}$, although there is no analytical solutions for the cohering power, they were showed to satisfy the additivity relation

$$
\begin{aligned}
&\mathrm{CP}_{l_1}(\mathcal{E}_1\otimes\mathcal{E}_2)+1=(\mathrm{CP}_{l_1}(\mathcal{E}_1)+1)(\mathrm{CP}_{l_1}(\mathcal{E}_2)+1),\\
&\mathrm{CP}_{r}(\otimes_{i=1}^{N}\mathcal{E}_i)=\sum_{i=1}^{N}\mathrm{CP}_{r}(\mathcal{E}_i).
\end{aligned}
\tag{450}
$$

Moreover, one may consider a slightly different definition of cohering power

$$
\mathrm{CP}^{(\rho)}(\mathcal{E})=\max_{\rho\in\mathcal{D}_{\mathcal{H}}}\{C(\mathcal{E}[\rho])-C(\rho)\},
\tag{451}
$$

which characterizes the maximum enhancement of quantum coherence after the action of the channel $\mathcal{E}$, it was found that for the 2-dimensional system, the two different cohering powers measured by the l_1 norm are always the same for any unitary channel, but when $d \geq 3$ or when the coherence is measured by the relative entropy, they can be different.

On the other hand, we know that the action of $\mathcal{E}$ on ρ can be implemented by an IO Λ on the product state of ρ and an ancillary state σ, i.e., $\Lambda(\rho \otimes \sigma) = \mathcal{E}(\rho) \otimes \sigma'$ see, e.g., the work of Baumgratz et al. (2014). Start from this point of view, Bu et al. (2017a) further gave an interpretation of the cohering power. To be explicit, they showed that the minimal amount of coherence of σ is just the cohering power of $\mathcal{E}$, i.e., $C(\sigma) \geq \mathrm{CP}(\mathcal{E})$.

Situ and Hu (2016) also explored the cohering and decohering powers of various typical channels, with however the coherence being measured by the relative entropy. These include the amplitude damping, phase damping, depolarizing, as well as the bit flip, bit-phase flip and phase flip channels. They also found that the cohering power can be enhanced by applying weak measurement and reversal operation to the qubit.

For the HS norm measure of quantum coherence, Zanardi et al. (2017) discussed the cohering power of the unitary and unital channels. As the HS norm is not a monotonic quantity under general CP maps, they restricted in their work only to those of the unital incoherent CP maps, under the action of which the HS norm of coherence is monotonically decreasing. By denoting $\Delta(\rho)$ the full dephasing of ρ in a given basis [see Eq. (142)] and $\tilde{\Delta} = \mathbb{1} - \Delta$ the complementary projection of Δ, they defined the cohering power of the quantum channel as the average coherence generated from a uniformly distributed incoherent states, i.e.,

$$C_{av}(\mathcal{E}) = \langle C_{l_2}(\mathcal{E}_{\mathrm{off}}[\psi])\rangle_{\psi}, \tag{452}$$

where $\mathcal{E}_{\mathrm{off}} = \tilde{\Delta}\mathcal{E}\Delta$, and the average is taken over the ensemble of pure states $\psi = |\psi\rangle\langle\psi|$ sampled randomly from the uniform Haar measure. That is to say, the uniform ensemble of incoherent states is generated by dephasing $\{\psi\}$.

When the channel is unitary, they found that the cohering power can be obtained analytically. For a d-dimensional system, it is given by

$$C_{av}(U) = \frac{1}{d+1}\left(1 - \frac{1}{d}\sum_{i,j} |\langle i|U|j\rangle|^4\right), \tag{453}$$

where the upper bound $C_{av}^{\max}(U) = (d-1)/d(d+1)$ is achieved when the bases $\{|i\rangle\}$ and $\{U|i\rangle\}$ are mutually unbiased, and the lower bound $C_{av}(U) = 0$ is achieved when U is an incoherent operation, i.e., $[U, \Delta] = 0$.

Similarly, when the channel $\mathcal{E}$ is unital, i.e., $\mathcal{E}(\mathbb{1}/d) = \mathbb{1}/d$, with $\{A_k\}$ being the corresponding Kraus operators, the cohering power is given by

$$C_{av}(\mathcal{E}) = \frac{1}{d(d+1)} \sum_{i,l\neq m} \left| \sum_k (A_k)_{li}(A_k)^*_{mi} \right|^2, \tag{454}$$

but now $C_{av}(\mathcal{E}) = 0$ does not always imply $[\mathcal{E}, \Delta] = 0$, that is to say, the cohering power $C_{av}(\mathcal{E})$ for unital channel is not faithful.

In fact, as the unitary operation U is a subset of the unital operation, $C_{av}(\mathcal{E})$ also covers the result of $C_{av}(U)$. Moreover, the above equation is equivalent to

$$C_{av}(\mathcal{E}) = \frac{1}{d+1}\{\mathrm{tr}[S\tilde{\omega}(\mathcal{E})] - \mathrm{tr}[S\omega(\mathcal{E})]\}, \tag{455}$$

where $\tilde{\omega}(\mathcal{E}) = \mathcal{E}^{\otimes 2}(\rho_{\mathrm{mm}})$, $\omega(\mathcal{E}) = (\Delta\mathcal{E})^{\otimes 2}[\rho_{\mathrm{mm}}]$, $\rho_{\mathrm{mm}} = \mathbb{1}/d$, and $S = \sum_{ij}|ij\rangle\langle ji|$ is the SWAP operator.

Styliaris et al. (2018) also examined power of the dephasing channel $\Delta_{B'}$ described by the projector $B' = \{|i'\rangle\langle i'|\}$ (i.e., $\Delta_{B'}[\rho] = \sum_{i'}\langle i'|\rho|i'\rangle|i'\rangle\langle i'|$) on generating quantum coherence defined with respect to the basis $B = \{|i\rangle\langle i|\}$. To be explicitly, they defined the cohering power as

$$C^B(\Delta_{B'}) = \int d\mu_{\mathrm{unif}}(\delta) C^B\left(\Delta_{B'}[\delta]\right), \tag{456}$$

where $d\mu_{\mathrm{unif}}(\delta)$ denotes the uniform measure in the $(d-1)$-dimensional simplex. $C^B(\Delta_{B'})$ is in fact the average coherence generated from uniformly distributed incoherent states $\delta \in \mathcal{I}$. When using the relative entropy of coherence as a quantifier, they proved that

$$C_r^B(\Delta_{B'}) = \tilde{Q}(X_U X_U^T) - \tilde{Q}(X_U), \tag{457}$$

and $C_r^B(\Delta_{B'}) = 0$ if and only if the dephasing operators $\Delta_{B'}$ and Δ_B commute. Here, X_U is bistochastic with elements $(X_U)_{ij} = |\langle i|U|j\rangle|^2$, U is the unitary operator ensures $|i'\rangle = U|i\rangle$, $\forall i$, and $\tilde{Q}(X) = \sum_j Q(\boldsymbol{p}_j)/d$, with $Q(\boldsymbol{p}_j)$ being the subentropy of the column vector $\boldsymbol{p}_j$ with elements $(\boldsymbol{p}_j)_i = (X_U)_{ij}$.

When one uses the HS norm of coherence, the power turns out to be (Styliaris et al., 2018)

$$C_{l_2}^B(\Delta_{B'}) = \frac{1}{d(d+1)}\mathrm{tr}[X_U X_U^T(1 - X_U X_U^T)], \tag{458}$$

and it is bounded from above by $(d-1)/4d(d+1)$, which is just one-quarter of the maximum $C_{av}^{\max}(U)$.

6.3.3. Coherence-breaking channels

Bu et al. (2016b) investigated coherence breaking channels (CBC) which were defined as those of the incoherent channels who destroy completely the coherence of any input state. They also discussed the selective CBC for which the Kraus operators $\{K_n\}$ give $K_n \rho K_n^\dagger \in \mathcal{I}$, and found that they are equivalent to CBC, i.e., the two sets $\mathcal{S}_{\rm cbc} = \mathcal{S}_{\rm scbc}$. The CBC are subsets of the entanglement-breaking channels (Horodecki et al., 2003b) and quantum–classical channels.

When a channel $\Phi \in \mathcal{S}_{\rm cbc}$, then $\Phi(|i\rangle\langle j|)$ is diagonal for any two incoherent basis states $|i\rangle$ and $|j\rangle$, and the action of Φ on ρ can be written as

$$\Phi(\rho) = \sum_i |i\rangle\langle i| \mathrm{tr}(\rho F_i), \tag{459}$$

with $\{F_i\}$ being the set of positive semidefinite operators satisfying $\sum_i F_i = \mathbb{1}$. For the special case of single-qubit state $\rho = (\mathbb{1} + \vec{r}\cdot\vec{\sigma})/2$, the channel

$$\Phi(\rho) = \frac{1}{2}[\mathbb{1} + (M\vec{r} + \vec{n})\cdot\vec{\sigma}], \tag{460}$$

belongs to CBC if the nonzero elements of M and $\vec{n}$ lie only in the third row of them.

Bu et al. (2016b) further introduced a notion which they termed as coherence-breaking index. It concerns the iterative actions of an incoherent quantum channel Φ on a given state, and can be defined explicitly as

$$n(\Phi) = \min\{n \geqslant 1 : \Phi^n \in \mathcal{S}_{\rm cbc}\}, \tag{461}$$

that is, $n(\Phi)$ characterizes the minimum number of iterations of Φ such that $\Phi^n(\rho) \in \mathcal{I}$ for any ρ. Clearly, if Φ is already a CBC, then $n(\Phi) = 1$, while for the case of $n(\Phi) = \infty$, Φ^n is not a CBC.

For the single-qubit case, they also investigated the sudden death phenomenon for the l_1 norm of coherence by using the result of Hu and Fan (2016a). Explicitly, the occurrence of coherence sudden death is only determined by forms of the incoherent channel and is independent of the initial state. But this does not apply to high dimensional states.

6.4. Evolution equation of quantum correlation and coherence

Referring to various measures of quantumness in open system, the search of certain dynamical law governing their evolution is of practical significance, as this can simplify the assessment of their robustness against decoherence. For entanglement measured by concurrence, its evolution was found to obey a factorization law for the initial two-qubit states and arbitrary quantum channels (Farías et al., 2009; Konrad et al., 2008), we review here the similar problems for geometric quantum correlations and quantum coherence.

6.4.1. Evolution equation of geometric quantum correlation

Similar to the evolution equation of entanglement measured by concurrence (Konrad et al., 2008), Hu and Fan (2015b) found that when a bipartite system traverses the local quantum channel, the evolution of GQD may demonstrate a factorization decay behavior.

For a bipartite state ρ decomposed as

$$\rho = -\frac{1}{d_A d_B}\mathbb{1}_{AB} + \rho_A \otimes \frac{1}{d_B}\mathbb{1}_B + \frac{1}{d_A}\mathbb{1}_A \otimes \rho_B + \rho_{co}, \tag{462}$$

where the reduced states $\rho_A = \mathrm{tr}_B\rho$, $\rho_B = \mathrm{tr}_A\rho$, and the traceless 'correlation operator' ρ_{co} is given by

$$\begin{aligned} &\rho_A = \frac{1}{d_A}\mathbb{1}_A + \vec{x}\cdot\vec{X}, \quad \rho_B = \frac{1}{d_B}\mathbb{1}_B + \vec{y}\cdot\vec{Y}, \\ &\rho_{co} = \sum_{i=1}^{d_A^2-1}\sum_{j=1}^{d_B^2-1} t_{ij} X_i \otimes Y_j, \end{aligned} \tag{463}$$

they proved that when the channel $\mathcal{E}$ gives $\mathcal{E}(\varrho) = q(t)\varrho$, with $\varrho = \vec{x}\cdot\vec{X}\otimes\mathbb{1}_B/d_B + \rho_{co}$, the evolution of $D_p(\mathcal{E}[\rho])$ fulfills the factorization decay behavior

$$D_p(\mathcal{E}[\rho]) = |q(t)|^p D_p(\rho), \tag{464}$$

which is solely determined by the product of the initial $D_p(\rho)$ and a channel-dependent factor $|q(t)|$, and

$$D_p(\rho) = \underset{\Pi^A \in \mathcal{M}}{\mathrm{opt}} \|\rho - \Pi^A(\rho)\|_p^p, \tag{465}$$

with opt representing the optimization over some class $\mathcal{M}$ of the local measurements $\Pi^A = \{\Pi_k^A\}$ acting on party A.

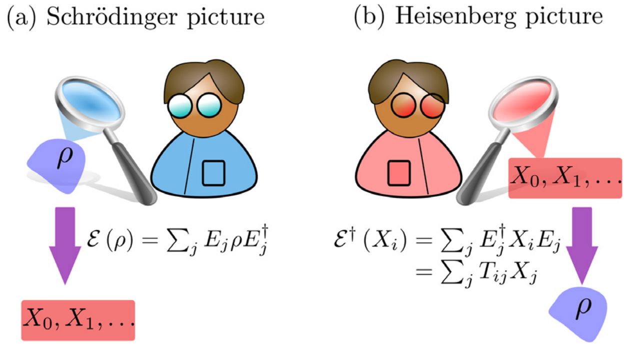

Fig. 6. The action of $\mathcal{E}$ on a state. In the Schrödinger picture (a), the state ρ evolves in time while the observables X_i are time-independent. But when we turn to the Heisenberg picture (b), the case is opposite, with which we obtain the condition for ρ obeying the evolution equation for quantum coherence.

By turning to the Heisenberg picture to describe the action of $\mathcal{E}$ (with the Kraus operators $\{E_j\}$), i.e., $\mathcal{E}^\dagger(X_i) = \sum_j E_j^\dagger X_i E_j$ (see Fig. 6), they identified the family of states for which the factorization relation holds. Explicitly, if $\mathcal{E}_1^\dagger(X_i) = q_A X_i$ for all $\{X_i\}$, and $\mathcal{E}_2^\dagger(Y_j) = q_B Y_j$ for all $\{Y_j\}$, then Eq. (464) holds for the families of ρ with

$$\begin{aligned} &(1)\ \text{arbitrary}\ \rho_A,\ \rho_B,\ \rho_{co}\ \ (\text{for}\ \mathcal{E}_1 \otimes \mathbb{1}_B), \\ &(2)\ \rho_A = \frac{1}{d_A}\mathbb{1}_A,\ \text{or}\ \rho_{co} = 0\ \ (\text{for}\ \mathcal{E}_1 \otimes \mathcal{E}_2\ \text{with}\ \mathcal{E}_2 \neq \mathbb{1}_B), \end{aligned} \tag{466}$$

while $\mathcal{E}_1^\dagger(X_k) = q_A X_k$ only for $\{X_k\}$ with $k = \{k_1, \ldots, k_\alpha\}$ ($\alpha < d_A^2 - 1$), and $\mathcal{E}_2^\dagger(Y_l) = q_B Y_l$ only for $\{Y_l\}$ with $l = \{l_1, \ldots, l_\beta\}$ ($\beta < d_B^2 - 1$), Eq. (464) holds for the families of ρ with

$$\begin{aligned} &(1)\ \rho_A = \rho_A^{(1)},\ \ \rho_{co} = \rho_{co}^{(1)}\ \ (\text{for}\ \mathcal{E}_1 \otimes \mathbb{1}_B), \\ &(2)\ \rho_A = \frac{1}{d_A}\mathbb{1}_A,\ \ \rho_{co} = \rho_{co}^{(2)},\ \ \text{or}\ \rho_{co} = 0\ (\text{for}\ \mathbb{1}_A \otimes \mathcal{E}_2), \\ &(3)\ \rho_A = \frac{1}{d_A}\mathbb{1}_A,\ \ \rho_{co} = \rho_{co}^{(3)},\ \ \text{or}\ \rho_A = \rho_A^{(1)},\ \ \rho_{co} = 0 \\ &\qquad (\text{for}\ \mathcal{E}_1 \otimes \mathcal{E}_2), \end{aligned} \tag{467}$$

where

$$\begin{aligned} &\rho_A^{(1)} = \frac{1}{d_A}\mathbb{1}_A + \sum_{k=k_1}^{k_\alpha} x_k X_k,\ \ \rho_{co}^{(1)} = \sum_{k=k_1}^{k_\alpha} \sum_{j=1}^{d_B^2-1} t_{kj} X_k \otimes Y_j, \\ &\rho_{co}^{(2)} = \sum_{i=1}^{d_A^2-1} \sum_{l=l_1}^{l_\beta} t_{il} X_i \otimes Y_l,\ \ \rho_{co}^{(3)} = \sum_{k=k_1}^{k_\alpha} \sum_{l=l_1}^{l_\beta} t_{kl} X_k \otimes Y_l. \end{aligned} \tag{468}$$

Besides the above statements, they also discussed the case of symmetric GQD, the families of states for which Eq. (464) holds are similar to those of Eqs. (467) and (468), and we do not list them here again.

6.4.2. Evolution equation of quantum coherence

For quantum coherence measured by l_1 norm, Hu and Fan (2016a) explored its evolution for a d-dimensional system traversing the quantum channel $\mathcal{E}$. As for any master equation which is local in time, whether Markovian, non-Markovian, of Lindblad form or not, one can always construct a linear map which gives $\rho(t) = \mathcal{E}(\rho(0))$ (the opposite case may not always be true), and the linear map can be expressed in the Kraus-type representations (Andersson et al., 2007). If the map $\mathcal{E}$ is CPTP, then one can explicitly construct the Kraus operators $\{E_\mu\}$ such that $\mathcal{E}(\rho) = \sum_\mu E_\mu \rho E_\mu^\dagger$.

For ρ of Eq. (154), one can turn to the Heisenberg picture to describe $\mathcal{E}$ via the map

$$\mathcal{E}^\dagger(X_i) = \sum_\mu E_\mu^\dagger X_i E_\mu, \tag{469}$$

which gives $x_i' = \mathrm{tr}(\rho \mathcal{E}^\dagger[X_i])$. As any Hermitian operator $\mathcal{O}$ on $\mathbb{C}^{d\times d}$ can always be decomposed as $\mathcal{O} = \sum_{i=0}^{d^2-1} r_i X_i$ ($r_i \in \mathbb{R}$), $\mathcal{E}^\dagger(X_i)$ can be further characterized by the transformation matrix T defined in Eq. (420), namely, $\mathcal{E}^\dagger(X_i) = \sum_{j=0}^{d^2-1} T_{ij} X_j$, where

$T_{ij} = \mathrm{tr}(\mathcal{E}^\dagger[X_i]X_j)/2$, and $X_0 = \sqrt{2/d}\mathbb{1}_d$. Clearly, $T_{00} = 1$, and $T_{0j} = 0$ for $j \geq 1$. This further gives

$$x_i' = \sum_{j=0}^{d^2-1} T_{ij}x_j. \tag{470}$$

By classifying state ρ of Eq. (154) into different families: $\rho = \{\rho^{\hat{n}}\}$, with $\rho^{\hat{n}} = \mathbb{1}_d/d + \chi\hat{n}\cdot\vec{X}/2$ ($\hat{n}$ is a unit vector in $\mathbb{R}^{d^2-1}$, and χ is smaller than $\sqrt{2(d-1)/d}$ as $\mathrm{tr}(\rho^{\hat{n}})^2 = |\chi|^2/2 + 1/d$), one can found that if $T_{k0} = 0$ for $k \in \{1, 2, \ldots, d^2 - d\}$, then the evolution of $C_{l_1}(\mathcal{E}[\rho^{\hat{n}}])$ obeys the factorization relation

$$C_{l_1}(\mathcal{E}[\rho^{\hat{n}}]) = C_{l_1}(\rho^{\hat{n}})C_{l_1}(\mathcal{E}[\rho_p^{\hat{n}}]), \tag{471}$$

with

$$\rho_p^{\hat{n}} = \frac{1}{d}\mathbb{1} + \frac{1}{2}\chi_p\hat{n}\cdot\vec{X}, \tag{472}$$

being the probe state, and $\chi_p = 1/\sum_{r=1}^{d_0}(n_{2r-1}^2 + n_{2r}^2)^{1/2}$.

As a corollary of the above equation, one can also show that if the operator $A = \sum_\mu E_\mu E_\mu^\dagger$ is diagonal, then the evolution of $C_{l_1}(\mathcal{E}[\rho^{\hat{n}}])$ is governed by Eq. (471). Moreover, if more restrictions are imposed on the quantum channel, e.g., if a channel $\mathcal{E}$ yields $\mathcal{E}^\dagger(X_k) = q(t)X_k$ for $\{X_k\}_{k=k_1,\ldots,k_\beta}$ ($\beta \leq d^2 - d$), with $q(t)$ containing information on $\mathcal{E}$'s structure, then

$$C_{l_1}(\mathcal{E}[\rho]) = |q(t)|C_{l_1}(\rho), \tag{473}$$

holds for all the

$$\rho = \frac{1}{d}\mathbb{1}_d + \frac{1}{2}\sum_{k=k_1}^{k_\beta} x_kX_k + \frac{1}{2}\sum_{l=d^2-d+1}^{d^2-1} x_lX_l. \tag{474}$$

The channel $\mathcal{E}$ satisfying the requirements includes the Pauli channel (covers bit flip, phase flip, bit-phase flip, phase damping, and depolarizing channels) and Gell-Mann channel given by Hu and Fan (2016a), and the generalized amplitude damping channel. They also constructed a quantum channel $\mathcal{E}_G$ for which $C_{l_1}(\mathcal{E}_G[\rho])$ obeys Eq. (473) for arbitrary initial state.

6.5. Preservation of GQD and quantum coherence

Along with the similar line for exploring quantum entanglement and entropic discord dynamics in open quantum systems, some works have also been devoted to the study of dynamical behaviors of various GQDs and coherence monotones. Apart from those focused on identifying freezing conditions discussed above, the others are aimed at seeking flexible methods to control their evolution. We summarize the key results in this section, mainly for the qubit states and typical noisy sources in quantum information processing.

Hu and Fan (2012b) discussed robustness of the HS norm of discord for two qubits coupled to a multimode vacuum electromagnetic field, and found that the robustness can be enhanced if appropriate local unitary operations were performed on the initial state of the system. Hu and Tian (2014) discussed trace norm of discord and Bures norm of discord for a two-qubit system subject to independent and common zero-temperature bosonic structured reservoirs. The results showed that the two GQDs can be preserved well or even be improved and generated by the noisy process of the common reservoir. If one can detune the transition frequency of the qubits to large enough values, the long-time preservation of these two GQDs in independent reservoirs can also be achieved. Moreover, it was found that the decay rates of GQD can be retarded apparently by properly choosing the Heisenberg type interaction of two qubits when they are embedded in two independent Bosonic structured reservoirs (Li et al., 2017).

If the noisy channel (or reservoir) coupled to the central system is non-Markovian, the backflow of information from the reservoir to the system can induce damped oscillation behaviors of the GQD. For two qubits subject to bosonic structured reservoirs with Lorentzian and Ohmic-like spectra, the relation between behaviors of GQDs and the extent of non-Markovianity of the reservoir have been studied (Hu and Lian, 2015). By analyzing their dependence on a factor whose derivative signifies the (non-)Markovianity of the dynamics, it was demonstrated that the non-Markovianity induced by the backflow of information from the reservoirs to the system enhances the GQDs in most of the parameter regions.

For a single qubit subjecting to pure dephasing channel with the Ohmic-like spectral densities, Zhang et al. (2015) compared the coherence evolution behaviors (measured by the l_1 norm and relative entropy) with different system and bath parameters. They found that the initial system–bath correlations are preferable for realizing long-lived coherence in the super-Ohmic baths, and the region of coherence trapping is enlarged with increasing the correlation parameter. For the given initial state with equal amplitudes, they obtained numerically the optimal Ohmicity parameter $\mu \doteq 1.46$ for the most efficient coherence trapping, which is independent of the coupling constant and the correlation parameter.

The atomic system is also an important candidate for various quantum information processing tasks. For the static polarizable two-level atoms interacting with a fluctuating vacuum electromagnetic field, Liu et al. (2016c) explored the coherence dynamics measured by the l_1 norm and relative entropy, both for an initial product state of the atoms and the field. The results show that for the initial one-qubit pure states and two-qubit Bell-diagonal states, the coherence cannot be protected for non-boundary electromagnetic field. Contrarily, when there is a reflecting boundary, the coherence will be trapped if the atom is close to the boundary and transversely polarized. The coherence can also be protected to some extent for other specific polarization directions. All these show that the coherence behavior is position and polarization dependent.

7. Quantum coherence and GQD in many-body systems

Quantum coherence can be regarded as a fundamental property in quantum realm. The many-body systems of condensed matter physics possess various quantum characteristics which may have no classical analog. In this sense, the exploring of quantum coherence in many-body systems may lead to some intriguing connections and may also result in developments in both research areas. We next start from an interesting concept in condensed matter physics.

7.1. Off-diagonal long-range order and l_1 norm of coherence

In theory of superconductivity, one of the well-known properties is the off-diagonal long-range order (ODLRO). Apparently, this property is related with l_1 norm quantum coherence measure, which uses the summation of the off-diagonal elements norm of a (reduced) density matrix quantifying coherence.

Let us consider a η-pairing state as an example. We start from the Hamiltonian of the Hubbard model,

$$\begin{aligned} H = &- \sum_{\sigma,\langle i,j\rangle} \left(c_{j,\sigma}^{\dagger} c_{k,\sigma} + c_{k,\sigma}^{\dagger} c_{j,\sigma} \right) \\ &+ U \sum_{j=1}^{L} \left(n_{j\uparrow} - \frac{1}{2} \right) \left(n_{j\downarrow} - \frac{1}{2} \right), \end{aligned} \tag{475}$$

where $\sigma = \uparrow, \downarrow$, and $\langle j, k\rangle$ is considered as a pair of the nearest-neighboring sites, $c_{j\sigma}^{\dagger}$ and $c_{j\sigma}$ are the creation and annihilation operators of fermions. The η-pairing operators at lattice site j are defined as

$$\eta_j = c_{j\uparrow} c_{j\downarrow}, \;\; \eta_j^{\dagger} = c_{j\downarrow}^{\dagger} c_{j\uparrow}^{\dagger}, \;\; \eta_j^z = -\frac{1}{2} n_j + \frac{1}{2}, \tag{476}$$

and they constitute a SU(2) algebra. The η operators are defined as $\eta = \sum \eta_j$ and $\eta^{\dagger} = \sum \eta_j^{\dagger}$. The η-pairing state is defined as (Essler et al., 1992; Fan and Lloyd, 2005; Yang, 1989)

$$|\Psi\rangle = (\eta^{\dagger})^N |\mathrm{vac}\rangle \tag{477}$$

where $|\mathrm{vac}\rangle$ is the vacuum state, and $|\Psi\rangle$ is an eigenstate of the Hubbard model. We can find that the η-pairing state is actually the completely symmetric state with N sites filled while the other $L - N$ sites unfilled up to a normalization factor. The ODLRO of this η-pairing state is shown as

$$C_{odlro} = \frac{\langle \Psi | \eta_k^{\dagger} \eta_l | \Psi \rangle}{\langle \Psi | \Psi \rangle} = \frac{N(L-N)}{L(L-1)}, \;\; k \neq l. \tag{478}$$

The off-diagonal element C_{odlro} is a constant which does not depend on the distance $|k - l|$, in particular when $|k - l| \to \infty$.

We may find that the density matrix $\rho = |\Psi\rangle\langle\Psi|$ of the η-pairing state is a L-qubit state, the quantity C_{odlro} corresponds to one off-diagonal element of ρ. The l_1 norm of η-pairing state can be calculated as

$$C_{l_1}(\rho) = L(L-1) C_{odlro}. \tag{479}$$

In this sense, the ODLRO is directly related with l_1 norm of coherence. As we mentioned that the coherence measure depends on a specified basis, here the definition of ODLRO naturally provides the basis by which the quantum coherence can be quantified.

By using this example, we try to show that further perspective study can be expected concerning about the quantum coherence in many-body systems.

7.2. Quantum coherence of valence-bond-solid state

Haldane conjectured that antiferromagnetic spin chains will be gapless for half-odd-integer spins and gapped for integer spins (Haldane, 1983a,b). The Affleck–Kennedy–Lieb–Tasaki (AKLT) model (Affleck et al., 1987, 1988) is a spin-1 chain in bulk and spin-1/2 at the two ends, which agrees with the Haldane conjecture. The ground state of AKLT model is known as the valence-bond-solid (VBS) state. The Hamiltonian of the AKLT model is written as

$$H = \sum_{j=1}^{N-1} \left(\vec{S}_j \cdot \vec{S}_{j+1} + \frac{1}{3}(\vec{S}_j \cdot \vec{S}_{j+1})^2 \right) + \pi_{0,1} + \pi_{N,N+1}, \tag{480}$$

where $\vec{S}$ is the spin-1 operator in bulk, π describes the interaction of spin-1 in bulk and spin-1/2 at one end.

The ground state, VBS state, is written as

$$|G\rangle = (\otimes_{j=1}^{N} P_{j\bar{j}})|\Psi^-\rangle_{\bar{0}1}|\Psi^-\rangle_{\bar{1}2}\cdots|\Psi^-\rangle_{\bar{N}N+1}, \tag{481}$$

where $|\Psi^-\rangle = (|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)/\sqrt{2}$ is a singlet state which corresponds to the operator form, $(a_{\bar{i}}^\dagger b_i^\dagger - b_{\bar{i}}^\dagger a_i^\dagger)|\text{vac}\rangle$, where $a^\dagger$ and $b^\dagger$ are bosonic creation operators. The projector P maps a two-qubit state, which is a four-dimensional Hilbert space, on a symmetric subspace which is three-dimension for spin-1 operator $\vec{S}$. So the VBS state is constructed by a chain of singlet states under the projection of P at the bulk sites and leaves the spin-1/2 at two ends (see Fig. 7).

By using the teleportation technique sequentially (Fan et al., 2004), the VBS state takes a form like the following

$$\begin{aligned} |G\rangle &= \frac{1}{3^{N/2}} \sum_{\alpha_j=1}^{3} |\alpha_1\rangle\cdots|\alpha_N\rangle \\ &\times \left[\mathbb{1} \otimes (\sigma_{\alpha_N}\cdots\sigma_{\alpha_1})\right]|\Psi^-\rangle_{\bar{0},N+1}, \end{aligned} \tag{482}$$

where $\mathbb{1}$ is the identity operator, $\sigma_1, \sigma_2, \sigma_3$ are Pauli matrices. It is proved that the reduced density matrix of continuously L bulk spins is invariant which does not depend on its position in the spin chain, it can be written as

$$\rho_L = \frac{1}{3}\sum_{\alpha,\alpha'} f_{\alpha\alpha'}|\alpha_1\rangle\langle\alpha_1'|\cdots|\alpha_L\rangle\langle\alpha_L'|, \tag{483}$$

where parameters $f_{\alpha\alpha'}$ can be determined as

$$f_{\alpha\alpha'} = \mathrm{tr}(\mathbb{1}\otimes V_\alpha)|\Phi^-\rangle\langle\Psi^-|(I\otimes V_{\alpha'}^\dagger), \tag{484}$$

with $V_\alpha = \sigma_{\alpha_L}\cdots\sigma_{\alpha_1}$, and $V_{\alpha'} = \sigma_{\alpha_L'}\cdots\sigma_{\alpha_1'}$.

Here we consider the measure of relative entropy of coherence. Due to the form of $f_{\alpha\alpha'}$, one may find that the diagonal matrix ρ_L^{diag} of ρ_L is a completely mixed state with tensor product of L identities and a normalization factor, so we have

$$S(\rho_L^{\mathrm{diag}}) = L\log_2 3. \tag{485}$$

This quantity corresponds to the volume quantity of the bulk L spins. On the other hand, we know (Fan et al., 2004) that the von Neumann entropy of ρ_L equals the von Neumann entropy of the state of two ends which is a Werner state

$$\begin{aligned} S(\rho_L) &= S(\tilde{\rho}_L), \\ \tilde{\rho}_L &= \frac{1}{4}(1-p_L)\mathbb{1} + p_L|\Psi^-\rangle\langle\Psi^-|, \end{aligned} \tag{486}$$

where $p_L = (-1/3)^L$. We know that the entropy of L bulk spins, $S(\rho_L)$, reaches to a constant 2 exponentially fast in terms of the number of bulk spins L.

With combination of those analyses, we can find that the relative entropy of coherence of bulk L spins,

$$C_r(\rho_L) = S(\rho_L^{\mathrm{diag}}) - S(\rho_L). \tag{487}$$

By substituting the results of Eqs. (485)–(486) into the definition, the coherence of the VBS state can be obtained straightforwardly. We would like to remark that this result is independent of the basis chosen because of the complete mixed form of ρ_L^{diag} is invariant for different bases.

For gapped one-dimensional system, the above results can be interpreted as,

$$C_r(\rho_L) = \text{volume-constant}. \tag{488}$$

Let us point out that the "constant" term corresponds to the area law (Eisert et al., 2010; Hamma et al., 2005).

Perspectively, it is worth exploring whether the relative entropy of coherence can be generally written as the difference between volume of the studied subsystem and the boundary term of area law,

$$C_r = \text{ volume} - \text{area law (boundary)}. \tag{489}$$

Further evidences of this expectation are necessary. Recently, it is shown that the volume effect can be found for the XY model, where the factor for the volume term is also important which can be used to distinguish different quantum phases (Wang et al., 2018b).

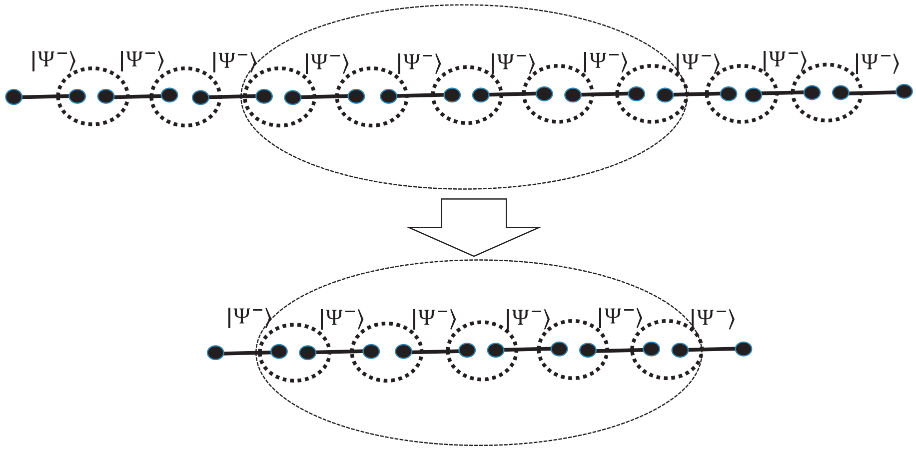

Fig. 7. Schematic picture of the VBS state for the AKLT model. The VBS state is constructed by a series of projections, represented as dashed circles, each acts on a pair of ends, represented as filled dots, of two singlet states. It can be shown that entropy of a subsystem of the VBS state, represented as large dashed circle, only depends on the size of the subsystems itself, so the VBS state can be chosen to contain the studied subsystem directly connected with two ends, as shown in lower part of the picture.

In the seminal work of Vidal et al. (2003), two different behaviors of entanglement entropy for one-dimensional gapped and gapless models were proposed. For gapless models like critical spin chains, the entanglement of the ground state for a bulk of spins grows logarithmically in number of particles in the bulk. The prefactor of the logarithm term is related with the central charge of the conformal field theory. The same scaling behavior holds also for mean entanglement at criticality for a class of strongly random quantum spin chains (Refael and Moore, 2004). For a gapped model, the entanglement approaches a constant bound. Additionally, the topological entanglement entropy and entanglement spectrum are studied based on the theory of entanglement (Hamma et al., 2005; Kitaev and Preskill, 2006; Levin and Wen, 2006; Li and Haldane, 2008). The entanglement entropy for a pure state is defined as the von Neumann entropy of the reduced density matrix of its subsystem. Rényi entropies parametrized by a parameter α are the generalizations of the von Neumann entropy. So topological entanglement entropy can also be generalized as topological entanglement Rényi entropies (Flammia et al., 2009). However, it is found that all topological Rényi entropies are the same, which is due to the fact that Rényi entropies are additive and the studied density matrix takes a product form. In correspondence, those results can be further explored from point of view of quantum coherence. There are some works about the characteristics of quantum phase transitions by quantum coherence, as presented next.

7.3. Quantum coherence and correlations of localized and thermalized states

Quantum dynamics of isolated quantum systems far from equilibrium has recently been extensively studied (Eisert et al., 2015). By principles of statistical mechanics, it is known that the non-equilibrium state will evolve to a thermalized state which is ergodic (Deutsch, 1991; Rigol et al., 2008; Srednicki, 1994), and no quantum correlation is expected to exist. For an isolated quantum system initially in a pure state, the time evolution is unitary transformation which keeps the system in a pure state. The thermalization means that the reduced density matrix of a subsystem, which is relatively small compared with the whole system, takes the form of a thermal state, $\rho^S = e^{-\beta H^S}/Z$, where β is the inverse of temperature, H^S is the Hamiltonian of the studied subsystem S, and $Z = \mathrm{tr}e^{-\beta H^S}$ is the partition function.

On the other hand, it is pointed out that the disorder may prevent the system from thermalizing, resulting in localized state. In general, there are two different types of localization, the single-particle localization in name of Anderson localization (Anderson, 1958) and the many-body localization (Basko et al., 2006; Gornyi et al., 2005). The many-body localization is induced by competition between interactions and disorder, in contrast, Anderson localization is only due to disorder but without interaction. Besides, thermalization cannot happen for integrable models because of the constraints imposed by infinite number of conserved quantities. There are various signatures in characterizing the thermalized states and localized states. Here we will review the properties of quantum coherence and quantum correlations for those states.

7.3.1. Entanglement entropy

The mechanism of thermalization is based on the eigenstate thermalization hypothesis, and the thermal state is ergodic (Altman and Vosk, 2015; Nandkishore and Huse, 2015a,b; Rigol et al., 2008). Those facts lead to that the thermalized state ρ^S takes a diagonal form, so state ρ^S possesses neither quantum coherence, nor quantum correlations. On the other hand, the von Neumann entropy of the thermal state, $S(\rho^S) = -\mathrm{tr}(\rho^S \log_2 \rho^S)$ satisfies the volume law, implying that it is proportional to the number of the particles L of the subsystem S. Considering that the isolated system is always in pure state, the von Neumann entropy of the reduced density matrix is the entanglement entropy itself. Thus in thermal phase, the entanglement entropy satisfies the volume law. If the initial state of the system is a highly excited state such as Néel state, the thermalized state approaches to the completely mixed state corresponding to infinite temperature, so entanglement entropy approaches L, corresponding to particle number of the subsystem for spin-1/2 particles. Note that L is the upper bound of the entanglement entropy.

The quantum dynamics of localizations, both Anderson localization and many-body localization, and thermalization can be well characterized by behaviors of entanglement entropy. Suppose the initial state is a product state like Néel state, which is also a highly excited state, the initial entropy will be zero for the subsystem S. For thermalization, the entropy will increase quickly and approaches its upper bound. For Anderson localization, similarly, the entropy will quickly saturate its bound but the bound is much smaller than that of the thermalized state. In contrast, many-body localization is a consequence of the competition between particle interactions and disorder. The localized state breaks the ergodicity and eigenstate thermalization hypothesis. The entanglement entropy does not obey the volume law. Instead, the entropy for the stationary state demonstrates a long time slow increase characterized as logarithmic increasing in time (Bardarson et al., 2012; Levi et al., 2016; Modak and Mukerjee, 2015; Nandkishore and Huse, 2015a,b; Ponte et al., 2015; Žnidarič et al., 2008), or algebraic with power-law interactions (Pino, 2014). Reminding that the coherence can be quantified as the diagonal entropy subtracting the von Neumann entropy, the coherence of the subsystem will demonstrate decrease for many-body localization.

Both Anderson localization and many-body localization have been realized experimentally. In system of trapped ions with long-range interactions, the growth of entanglement is shown by measuring the quantum Fisher information (Smith et al., 2016). Recently, the entanglement entropy logarithmic increase in time of many-body localization is successfully demonstrated in a 10-qubit superconducting quantum simulation based on single-shot state tomography measurement (Xu et al., 2018). The many-body localization and thermalization can also be distinguished by energy spectrum of the system. In thermal phase, the energy levels of the system tend to repel one another and their statistics are Wigner–Dyson distribution, while for many-body localized state, the energy levels show a Poisson statistics (Atas et al., 2013; Bohigas et al., 1984; Oganesyan and Huse, 2007). These phenomena are also demonstrated experimentally (Roushan et al., 2017).

7.3.2. *Entanglement spectrum*

The entanglement entropy of a ground state is just one quantity based on entanglement spectrum (Li and Haldane, 2008) due to Schmidt decomposition for a pure state. The entanglement spectrum possesses more information which may be invisible for entanglement entropy. For a ground state $|G\rangle$, the reduced density matrix for a bulk of L sites is written as ρ_L, which can be rewritten as

$$\rho_L = e^{-H_{\mathrm{E}}}. \tag{490}$$

The entanglement spectrum is the energy spectrum of the so-defined entanglement Hamiltonian H_{E}. It is shown that there is one-to-one correspondence between low-energy edge states of the system with open boundary condition and the low-lying eigenstates of the entanglement Hamiltonian (Fidkowski, 2010; Qi et al., 2012).

The entanglement spectrum of the ground state for a topological Chern insulator with disorder exhibits level repulsion, which is consistent with Wigner–Dyson distribution. This result in addition with energy spectrum and Chern number can be used to describe transition of Chern insulator to Anderson-insulator (Prodan et al., 2010). The many-body localization and thermalization and the periodically driven systems, which are known as Floquet systems, can be characterized by entanglement spectrum (Geraedts et al., 2016). The level statistics of the entanglement spectrum in the thermalizing phase is governed by an appropriate random matrix ensemble. The Floquet entanglement spectrum has similar results showing a result beyond eigenstate thermalization hypothesis. In many-body localized phase, the entanglement spectrum shows level repulsion and obeys a semi-Poisson distribution. Also, the dynamical many-body localization is observed in an integrable system with periodically driven (Keser et al., 2016). Perspectively, the study of quantum benchmarks such as quantum coherence, entanglement may be performed for those systems and phases like Floquet topological insulators induced by disorder (Titum et al., 2015).

7.4. *Quantum coherence and quantum phase transitions*

Quantum phase transition (QPT) describes an abrupt change for properties of the ground state of a many-body system driven by its quantum fluctuations. It is a purely quantum process and is caused by variation of the system parameters of the Hamiltonian, such as the spin coupling and external magnetic field (Sachdev, 2000). With the development of quantum entanglement theory, it is natural to study QPT of a many-body system from the point of view of entanglement. Indeed, it has been found that the singularity and extreme point of entanglement or its derivative can be used for detecting QPTs. An overview for the related progress can be found in the work of Amico et al. (2008) and Zeng et al. (2015).

As quantum coherence measures defined within the framework of Baumgratz et al. (2014) are also quantitative characterizations of the quantum feature of a system, they are hoped to play a role in studying quantum phase transitions (QPTs) of the many-body systems. We review briefly in this section some main progress for such studies.

Please cite this article in press as: Hu M.-L., et al., Quantum coherence and geometric quantum discord. Physics Reports (2018), https://doi.org/10.1016/j.physrep.2018.07.004.

Chen et al. (2016a) demonstrated role of the coherence susceptibility on studying QPTs at both absolute zero and finite temperatures. Here, the coherence susceptibility is defined as the first derivative of the relative entropy of coherence $C_r(\rho)$, that is,

$$\chi^{\rm co} = \frac{\partial C_r(\rho)}{\partial \lambda}, \tag{491}$$

with λ being a characteristic parameter of the system Hamiltonian. For the transverse Ising model with the Hamiltonian H_I, the spin-1/2 Heisenberg XX model with $\hat{H}_{XX}$, and the Kitaev honeycomb model with $\hat{H}_K$, described by

$$\begin{aligned} \hat{H}_I &= -\sum_{i=1}^{N} \sigma_i^z \sigma_{i+1}^z - \lambda \sum_{i=1}^{N} \sigma_x, \\ \hat{H}_{XX} &= -\frac{1}{2} \sum_{i=1}^{N} (\sigma_i^x \sigma_{i+1}^x + \sigma_i^y \sigma_{i+1}^y) - \lambda \sum_{i=1}^{N} \sigma_i^x, \\ \hat{H}_K &= -\sum_{\alpha=\{x,y,z\}} J_\alpha \sum_{i,j\in\alpha-\text{links}} \sigma_i^\alpha \sigma_j^\alpha, \end{aligned} \tag{492}$$

where λ is the strength of the external magnetic field in units of the interaction energy, they showed that apart from the figure of merit that this method requires no prior knowledge of order parameter (the same as those based on entanglement and discord), the coherence susceptibility pinpoints not only the exact QPT points via its singularity with respect to λ, but also the temperature frame of quantum criticality. In particular, the latter has been considered to be a superiority of the coherence susceptibility method.

Karpat et al. (2014) showed validity of the Skew-information-based coherence measure $\mathrm{I}(\rho, K)$ (will be called the K coherence for brevity) and its lower bound $\mathrm{I}^L(\rho, K)$ on studying QPTs (Girolami, 2014). They considered the spin-1/2 Heisenberg XY model described by the Hamiltonian

$$\begin{aligned} \hat{H} = &-\frac{\lambda}{2} \sum_i [(1+\gamma)\sigma_i^x \sigma_{i+1}^x + (1-\gamma)\sigma_i^y \sigma_{i+1}^y] \\ &- \sum_i \sigma_i^z, \end{aligned} \tag{493}$$

with $0 \le \gamma \le 1$ being the anisotropy parameter, and λ strength of the inverse magnetic field. They calculated the single-spin coherence $\mathrm{I}(\rho, \sigma^\beta)$, two-spin local coherence $\mathrm{I}(\rho, \sigma^\beta \otimes \mathbb{1}_2)$ $(\beta = x, y, z)$, and their lower bounds. The numerical results show that the divergence of the first derivatives of $\mathrm{I}(\rho, \sigma^x)$ and $\mathrm{I}(\rho, \sigma^{x,y,z} \otimes \mathbb{1}_2)$ (including their lower bounds) with respect to γ pinpoint exactly the transition point $\gamma_c = 1$ [$\mathrm{I}(\rho, \sigma^y \otimes \mathbb{1}_2)$ fails for the special case $\gamma = 0.5$], while the derivatives of $\mathrm{I}(\rho, \sigma^{x,y,z} \otimes \mathbb{1}_2)$ also detect the factorization point $\lambda_f \sim 1.1547$. Moreover, the performance of $\mathrm{I}^L(\rho, \sigma^x)$ in detecting QPTs at relatively high temperatures outperforms that of $\mathrm{I}(\rho, \sigma^x)$ for the considered model. A review of these results in addition with some quantum correlations is presented in (Cakmak et al., 2015).

Lei and Tong (2016) studied quantum coherence measured by the WY skew information on diagnosing critical points of the spin-1/2 transverse field XY model with the $XZY-YZX$ type of three-spin interactions. The Hamiltonian is given by

$$\begin{aligned} \hat{H} = -\sum_{i=1}^{N} \Bigg[&\frac{1+\gamma}{2} \sigma_i^x \sigma_{i+1}^x + \frac{1-\gamma}{2} \sigma_i^y \sigma_{i+1}^y + h\sigma_l^z \\ &+ \frac{\alpha}{4} (\sigma_{i-1}^x \sigma_i^z \sigma_{i+1}^y - \sigma_{i-1}^y \sigma_i^z \sigma_{i+1}^x) \Bigg]. \end{aligned} \tag{494}$$

By examining the single-spin $\sigma^{x,y,z}$ coherence [i.e., $K = \sigma^x, \sigma^y$, or σ^z in Eq. (75)], the two-spin local $\sigma^{x,y,z}$ coherence [i.e., $K = \sigma^x \otimes \mathbb{1}_2, \sigma^y \otimes \mathbb{1}_2$, or $\sigma^z \otimes \mathbb{1}_2$], and their lower bounds $\mathrm{I}^L(\rho, K)$ (Girolami, 2014), Lei and Tong (2016) found that if the three-spin interaction $\alpha = 0$ and the external magnetic field $h < 1$, the single-spin $\sigma^{x,y,z}$, two-spin local σ^z coherence, and their lower bounds are extremal at the critical point $\gamma_c = 0$ of anisotropy transition. But the two-spin local σ^x (σ^y) coherence and its lower bound decrease (increase) with the increasing γ. Their first derivatives with respect to γ are minimal (maximum) at the critical point $\gamma_c = 0$, and show scaling behaviors with respect to $\log N$, i.e., $d\mathrm{Q}/d\gamma = a_1 + a_2 \log_2 N$, where $\mathrm{Q} = \mathrm{I}(\rho, K)$ or $\mathrm{I}^L(\rho, K)$, and a_1 and a_2 are the system-dependent parameters.

When the three-spin interaction is introduced, there will be a gapless phase in the range $h \in [h_{c1}, h_{c2}]$ for $\gamma < \alpha$. The system undergoes two QPTs (second order transitions) with increasing h, the first from gapped phase to gapless phase when h increases from $h < h_{c1}$ to $h > h_{c1}$, and the second from the gapless phase to gapped phase when h increases from $h < h_{c2}$ to $h > h_{c2}$. For this case, it was found that both the single-spin and two-spin local $\sigma^{x,y,z}$ coherence and their lower bounds are affected by the existence of α only in the gapless phase, and the two critical points h_{c1} and h_{c2} of the gapless phase can be pinpointed by the extremal points of their first derivatives. But different from that of $\alpha = 0$, there is no size effect of the corresponding derivatives of coherence around the critical points, that is, the derivatives for different N are almost the same.

Li and Lin (2016) also showed effectiveness of the two-spin local σ^β ($\beta = x, y, z$) coherence $\mathrm{I}^L(\rho, \sigma^\beta)$ in detecting QPTs of different physical systems. For the XY model with transverse magnetic fields and $XZX+YZY$ type of three-spin interactions [the Hamiltonian is similar as that of Eq. (494), with only the terms in the second line being replaced by $\alpha(\sigma_{i-1}^x\sigma_i^z\sigma_{i+1}^x + \sigma_{i-1}^y\sigma_i^z\sigma_{i+1}^y)$]. Contrary to the case of $h = 0.5$ studied in Lei and Tong (2016), when $h = \alpha = 0$, it was found that while the extremal of $\mathrm{I}^L(\rho, \sigma^z \otimes \mathbb{1}_2)$ can pinpoint the critical points of QPT at $\gamma_c = 0$, $\mathrm{I}^L(\rho, \sigma^x \otimes \mathbb{1}_2)$ [$\mathrm{I}^L(\rho, \sigma^y \otimes \mathbb{1}_2)$] increases (decreases) with γ, and its first derivative with respect to γ is maximal (minimal) at the first-order QPT point $\gamma_c = 0$. For the second-order QPT at $h_c = 1$, the derivative of $\mathrm{I}^L(\rho, \sigma^x \otimes \mathbb{1}_2)$ with respect to h at $h_c = 1$ shows a size-dependent scaling behavior, which implies that it will be divergent in the thermodynamic limit. Even at finite temperature, $\mathrm{I}^L(\rho, \sigma^x \otimes \mathbb{1}_2)$ and its first derivative can also detect the second order QPT at $\alpha = 0.5$ for $\gamma = 0.5$ and $h = 0$. Moreover, they also showed that the two-spin local coherence can detect QPTs for the one-dimensional half-filled Hubbard model with both on-site and nearest-neighboring interactions and topological phase transition for the Su–Schrieffer–Heeger model.

The amount of WY skew information $\mathrm{I}(\rho, K)$ is determined by the observable one chooses. Luo (2003) introduced a quantity

$$Q(\rho) = \sum_i \mathrm{I}(\rho, X_i) \tag{495}$$

where $\{X_i\}$ is the set of observables which constitute an orthonormal basis, and proved it to be independent of the choice of $\{X_i\}$ (Luo, 2006). Based on this, Cheng et al. (2015) explored its role in detecting critical points of QPTs and the factorization transition of the spin model. For the density matrix ρ^{AB} of two neighboring spins, they calculated

$$F(\rho^{AB}) = Q_A(\rho^{AB}) - Q_A(\rho^A \otimes \rho^B), \tag{496}$$

with $Q_A(\rho^{AB}) = \sum_i \mathrm{I}(\rho^{AB}, X_i \otimes \mathbb{1}_B)$, and likewise for $Q_A(\rho^A \otimes \rho^B)$. For the XY model with transverse magnetic fields, their results show that the second-order QPT from the ferromagnetic to the paramagnetic phase (Ising transition) can be detected by the minimum of the first derivative of $F(\rho^{AB})$ with respect to h. At the vicinity of the transition point $h_c = 1$, $\partial F(\rho^{AB})/\partial h$ shows a size-dependent scaling behavior, and is logarithmic divergent in the thermodynamic limit. On the other hand, the first-order transition from a ferromagnet with magnetization in the x direction to one with magnetization in the y direction (anisotropy transition) at $\gamma_c = 0$ can be detected directly by the minimum of $F(\rho^{AB})$, but its first derivative with respect to γ is continuous and size-independent. Moreover, it was found that both $\partial F(\rho^{AB})/\partial h$ and $\partial F(\rho^{AB})/\partial \gamma$ are discontinuous along the curve $h^2 + \gamma^2 = 1$. The emergence of the discontinuity pinpoints the factorization transition for ground states of the considered system, and has its roots in the elements of $\rho^{1/2}$.

Compared with the spin-1/2 models, various high-spin systems show richer phase diagrams. Malvezzi et al. (2016) considered the spin-1 XXZ model and bilinear biquadratic model, with the Hamiltonian

$$\begin{aligned} \hat{H}_{XXZ} &= \sum_i (S_i^x S_{i+1}^x + S_i^y S_{i+1}^y + \Delta S_i^z S_{i+1}^z), \\ \hat{H}_{BB} &= \sum_i [\cos\theta(\mathbf{S}_i \cdot \mathbf{S}_{i+1}) + \sin\theta(\mathbf{S}_i \cdot \mathbf{S}_{i+1})^2] \end{aligned} \tag{497}$$

where $\mathbf{S}_i = (S_i^x, S_i^y, S_i^z)$ are spin-1 operators. For the spin-1 XXZ model, the relative entropy and l_1 norm of coherence for two neighboring spins, and the local two-spin S^x and S^y coherence cannot detect the Kosterlitz–Thouless QPT at $\Delta_{c2} \approx 0$, while their inflection points detect the Ising type second-order QPT at $\Delta_{c2} \approx 1.185$. Moreover, the extremum of single-spin S^x coherence pinpoints the SU(2) symmetry point $\Delta = 1$. For the spin-1 bilinear biquadratic model, the single spin density matrix is diagonal in S_z basis for all values of the anisotropy parameter, so all coherence measures are zero. On the other hand, the transition point can be identified by mutual information and discord, which coincidences to both the infinite order Kosterlitz–Thouless transition and the SU(3) symmetry point $\theta = 0.25\pi$.

7.5. GQD and quantum phase transition

The singularity or extreme point of QD can be used for detecting QPTs. Werlang et al. (2010) studied one such problem. They considered a general Heisenberg XXZ model with the Hamiltonian

$$\hat{H} = J \sum_{i=1}^{N} (\sigma_i^x \sigma_{i+1}^x + \sigma_i^y \sigma_{i+1}^y + \Delta \sigma_i^z \sigma_{i+1}^z), \tag{498}$$

and by setting $J = 1$, they calculated QD of Eq. (5) as well as its first and second order derivatives for thermal states of the neighboring spins, and showed that it can efficiently detect the QPT points $\Delta = \pm 1$ for this model even at finite temperature, while the entanglement measured by entanglement of formation does not. This shows potential role of QD in investigating QPT. In particular, it is very important for experimental characterization of QPTs as in principle one is unable to reach a zero temperature experiment.

Dillenschneider (2008) studied QD for ground states of the transverse Ising and Heisenberg XXZ model, and found that the amount of QD increases close to the QPT points. Indeed, there are many other related works discussing role of the QD defined in Eq. (5) in detecting QPTs, and we refer to the work of Modi et al. (2012) for a detailed overview in this respect. In what follows, we focus on role of GQDs on studying QPTs in various many-body systems.

Paula et al. (2013b) examined ground state properties of the Heisenberg XXZ model Eq. (498) by setting $J = -1$. By employing the trace norm of discord as a quantifier of correlation, they found that $D_T(\rho)$ defined in Eq. (33) as well as $C_T(\rho)$ and $T_T(\rho)$ defined in Eq. (41) detects successfully the first-order phase transition at $\Delta = 1$. On the other hand, the infinite-order QPT at $\Delta = -1$ can only be detected by the classical correlation $C_T(\rho)$, while $D_T(\rho)$ and $T_T(\rho)$ failed. This seems to casting a doubt on the usefulness of GQD, but for other many-body systems it may work effectively for detecting phase transition points.

By using the quantum renormalization group method, Song et al. (2014) studied HS norm of discord for ground states of the Heisenberg XXZ model with Dzyaloshinskii–Moriya (DM) interaction. The Hamiltonian reads

$$\begin{aligned}\hat{H} = \frac{J}{4}\sum_{i=1}^{N}[\sigma_i^x\sigma_{i+1}^x + \sigma_i^y\sigma_{i+1}^y + \Delta\sigma_i^z\sigma_{i+1}^z \\ + D(\sigma_i^x\sigma_{i+1}^y - \sigma_i^y\sigma_{i+1}^x)].\end{aligned} \tag{499}$$

Their calculation shows that the HS norm of discord can effectively characterize the QPT point $\Delta_c = \sqrt{1+D^2}$ separating the spin–fluid phase and the Néel phase.

Huang et al. (2017) also studied quantum correlations for the ground state properties of several three different spin models, but they used the trace norm of discord. First, for the XXZ model given in Eq. (499), they found that the trace norm of discord detects successfully the QPT point Δ_c. Second, for the Ising model with DM interaction,

$$\hat{H} = \frac{J}{4}\sum_{i=1}^{N}[\sigma_i^z\sigma_{i+1}^z + D(\sigma_i^x\sigma_{i+1}^y - \sigma_i^y\sigma_{i+1}^x)], \tag{500}$$

it was shown that the trace norm of discord can also be used to detect the critical point $D = 1$ which separates the antiferromagnet phase and chirality phase. Huang et al. (2017) also considered the Heisenberg XXZ model with staggered DM interaction, with the Hamiltonian being given by

$$\begin{aligned}\hat{H} =& \frac{J}{4}\sum_{i=1}^{N}[(1+\Delta)\sigma_i^x\sigma_{i+1}^x - (1-\Delta)\sigma_i^y\sigma_{i+1}^y \\ &+ D(\sigma_i^x\sigma_{i+1}^y + \sigma_i^y\sigma_{i+1}^x)],\end{aligned} \tag{501}$$

and their result showed that the trace norm of discord also detects successfully the region $|\Delta| \leq \sqrt{1+D^2}$ in which the system is in the Néel phase. Concerning entanglement properties of this model, we refer to the work of Ma et al. (2011).

For the XX model with transverse magnetic field as showed in Eq. (492), Cheng et al. (2016) found that the trace norm of discord can also effectively characterize the second-order QPT occurs at $\lambda_c = 1$ which separates the ferromagnetic and paramagnetic phases. Cheng et al. (2017) studied QPT in an Ising-XXZ diamond model. By analyzing scaling behavior of the trace norm of discord for the thermal state, they found that around the critical lines, its first-order derivative exhibits a maximal at finite temperature and diverges when $T \to 0$.

Moreover, Filho et al. (2017) studied the problem of many-body localization (MBL) in a spin-1/2 Heisenberg model with random on-site disorder of strength h. The Hamiltonian is

$$\hat{H} = \frac{1}{2}\sum_{i=1}^{N}[J(\sigma_i^x\sigma_{i+1}^x + \sigma_i^y\sigma_{i+1}^y + \sigma_i^z\sigma_{i+1}^z) + h_i\sigma_i^z], \tag{502}$$

where h_i are uniformly distributed random numbers in the interval $[-h, h]$. They found that the derivatives of the trace norm of discord of Eq. (33) and the geometric classical and total correlations of Eq. (41) give the range $h_c/J \in [3, 4]$ for the MBL critical point. This estimate is in accordance with the result $h_c/J \sim 3.8$ given in the literature (Goold et al., 2015; Luitz et al., 2015).

8. Quantum correlations and coherence in relativistic settings

Since the early 20th century, much efforts have been put forward to bridge the gap between quantum mechanics and relativity theory, which are two fundamentals of modern physics. The reconciliation between them gives birth for quantum field theory (QFT), and several predictions have been made based on this theory. A fundamental prediction in QFT is that the particle content of a quantum field is observer dependent, such a phenomenon is named Unruh effect (Unruh, 1976; Crispino et al., 2008). Again, the phenomenon of a quantum field is in vacuum state as observed by a freely falling observer of an eternal black hole, while it is a thermal state for an observer who hovers outside the event horizon of the black hole. Such a phenomenon is named Hawking effect. The study of quantum correlation in a relativistic framework is not only helpful to understand some of the key questions in quantum information theory, but also plays an important role in the black hole entropy and black hole information paradox (Hawking, 1976; Terashima, 2000). Following the pioneering work of Peres et al. (2002), many authors have studied quantum correlations in relativistic setting from different aspects.

8.1. Quantum correlations for free field modes

For a free mode scalar field, the dynamics of the field obeys the Klein–Gordon (KG) equation (Birrell and Davies, 1982)

$$\frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^{\mu}}\left(\sqrt{-g}g^{\mu\nu}\frac{\partial\phi}{\partial x^{\nu}}\right)=0, \tag{503}$$

where g is the determinant of the metric $g_{\mu\nu}$ (Wald, 1994). Similarly, the motion equation of a Dirac field Ψ in a background reads

$$i\gamma^{\mu}(x)\left(\frac{\partial}{\partial x^{\mu}}-\Gamma_{\mu}\right)\Psi=m\Psi, \tag{504}$$

where the background-dependent Dirac matrices $\gamma^{\mu}(x)$ relate to the matrices in flat space through $\gamma^{\mu}(x)=e_a^{\mu}(x)\bar{\gamma}^a$, and $\bar{\gamma}^a$ are the flat-space Dirac matrices. Here,

$$\Gamma_{\mu}=\frac{1}{8}[\gamma^{\alpha},\gamma^{\beta}]e_{\alpha}^{\nu}e_{\beta\nu;\mu}, \tag{505}$$

are the spin connection coefficients. Throughout this section we set $G=c=\hbar=\kappa_B=1$.

The field (either scalar field or Dirac field) can be quantized in terms of a complete set of modes $u_k(x,\eta)$, which is an orthonormal basis of solutions of the scalar field (or Dirac field). That is,

$$\Phi(\Psi)=\int d^3k(a_k u_k+a_k^{\dagger}u_k^*), \tag{506}$$

where Φ denotes the scalar field and Ψ denotes the Dirac field, k is the wave vector labeling the modes and for massless fields $\omega=|k|$.

For a scalar field, the positive and negative frequency modes satisfy the canonical commutation relations $[a_k,a_{k'}^{\dagger}]=\delta^3(k-k')$, while for the Dirac fields the anti-commutation relations $\{a_k,a_{k'}^{\dagger}\}=\delta^3(k-k')$ should be satisfied. The annihilation operators a_k define the vacuum state $|0\rangle$ through $a_k|0\rangle=0,\ \forall k$. A different inequivalent choice of modes $\{\tilde{u}_k\}$ might exist which satisfies the same equation of motion in different spacetime background. For example, the appropriate coordinates to describe the accelerated observer's motion is the Rindler coordinates (η,ξ), which is given by the transformation

$$t=a^{-1}e^{a\xi}\sinh(a\eta),\ \ x=a^{-1}e^{a\xi}\cosh(a\eta). \tag{507}$$

Solving the KG equation or Dirac equation in the Rindler coordinates, we obtain some sets of positive frequency modes propagating in the regions I and II of the Rindler spacetime, respectively. For free scalar fields, the positive frequency modes can be used to expand the field as (Fuentes-Schuller and Mann, 2005)

$$\Phi=\int d\omega[\hat{a}_{\omega}^{I}\Phi_{\omega,\mathrm{I}}^{+}+\hat{b}_{\omega}^{I\dagger}\Phi_{\omega,\mathrm{I}}^{-}+\hat{a}_{\omega}^{II}\Phi_{\omega,\mathrm{II}}^{+}+\hat{b}_{\omega}^{II\dagger}\Phi_{\omega,\mathrm{II}}^{-}], \tag{508}$$

where $\hat{a}_{\omega}^{I}$ and $\hat{b}_{\omega}^{I\dagger}$ are the bosonic annihilation and anti-boson creation operators in the Rindler region I, and $\hat{a}_{\omega}^{II}$ and $\hat{b}_{\omega}^{II\dagger}$ are the bosonic annihilation and creation operators in the region II.

The quantum field theory for Dirac fields is constructed by expanding the field in terms of the positive and negative frequency modes (Alsing et al., 2006)

$$\Psi=\int d\mathbf{k}[\hat{c}_{\mathbf{k}}^{I}\Psi_{\mathbf{k}}^{I+}+\hat{d}_{\mathbf{k}}^{I\dagger}\Psi_{\mathbf{k}}^{I-}+\hat{c}_{\mathbf{k}}^{II}\Psi_{\mathbf{k}}^{II+}+\hat{d}_{\mathbf{k}}^{II\dagger}\Psi_{\mathbf{k}}^{II-}], \tag{509}$$

where $\hat{c}_{\mathbf{k}}^{I}$ and $\hat{d}_{\mathbf{k}}^{I\dagger}$ are the fermionic annihilation and creation operators acting on the state in region I, and $\hat{c}_{\mathbf{k}}^{II}$ and $\hat{d}_{\mathbf{k}}^{II\dagger}$ are the fermionic operators in the region II. The above positive and negative frequency modes are defined in terms of the future-directed timelike Killing vector in different regions, in Rindler region I the Killing vector is ∂_{η} and in the region II the Killing vector is $\partial_{-\eta}$.

After some calculations, the Minkowski vacuum is found to be an entangled two-mode squeezed state for a free scalar field

$$|0_{\omega}\rangle_{\mathrm{M}}=\frac{1}{\cosh r_{\omega}^2}\sum_{n=0}^{\infty}\tanh r_{\omega}^{n}|nn\rangle_{\omega}, \tag{510}$$

where $\cosh r=(1-e^{-2\pi\omega/a})^{-1/2}$ and a is Rob's acceleration. For a free Dirac field, the Minkowski vacuum has the following form

$$|0_{\mathbf{k}}\rangle_{M}=\cos r|0_{\mathbf{k}}\rangle_{I}|0_{-\mathbf{k}}\rangle_{II}+\sin r|1_{\mathbf{k}}\rangle_{I}|1_{-\mathbf{k}}\rangle_{II}, \tag{511}$$

where $\cos r=(e^{-2\pi\omega/a}+1)^{-1/2}$.

Please cite this article in press as: Hu M.-L., et al., Quantum coherence and geometric quantum discord. Physics Reports (2018), https://doi.org/10.1016/j.physrep.2018.07.004.

8.1.1. Quantum entanglement

Fuentes-Schuller and Mann (2005) studied quantum entanglement between two free bosonic modes as observed by two relatively accelerated observers. They found that the quantum entanglement is an observer-dependent quantity in noninertial frames. A maximally entangled initial state in an inertial frame becomes less entangled under the influence of the Unruh effect. In the infinite acceleration limit, the distillable entanglement for the final state of the scalar field vanishes. Alsing et al. (2006) studied the entanglement between two free modes of a Dirac field in noninertial frames. They found that entanglement between the Dirac modes is destroyed by the Unruh effect. Differently, the entanglement of the fermionic modes asymptotically reaches a nonzero minimum value in the infinite acceleration limit.

Ling et al. (2007) studied entanglement of the electromagnetic field in a noninertial reference frame. They employed the photon helicity entangled state and found that the logarithmic negativity of the final state remains the same as those in the inertial reference frame, which is completely different from that of the particle number entangled state. Pan and Jing (2008a) investigated the entanglement between two modes of free scalar and Dirac fields. They proved that the different behavior of the field modes is owing to the in equivalence of the quantization of the free field modes in the Minkowski and the Rindler coordinates. In the infinite-acceleration limit, the mutual information equals the half mutual information of the initial state, which is independent of the initial state and the type of field.

Adesso et al. (2007) studied the distribution of entanglement between modes of a free scalar field from the perspective of observers in uniform acceleration. We consider a two-mode squeezed state of the field from an inertial perspective, and analytically study the degradation of entanglement due to the Unruh effect, in the cases of either one or both observers undergoing uniform acceleration. The effect of Unruh effect on a quantum radiation can be described by a two-mode squeezing operator acting on the input state of the quantum system. In the phase space the symplectic phase-space representation, $S_{B,\bar{B}}(r)$ for the two-mode squeezing transformation is (Adesso et al., 2007)

$$S_{B,\bar{B}}(r)=\begin{pmatrix}\cosh r\mathbb{1}_2 & \sinh rZ_2\\ \sinh rZ_2 & \cosh r\mathbb{1}_2\end{pmatrix}, \tag{512}$$

where $\mathbb{1}_2$ is a 2×2 identity matrix and $Z_2=\mathrm{diag}\{1,-1\}$. After the transformation, the final state of the entire three-mode system is given by the covariance matrix (Adesso et al., 2007)

$$\begin{aligned}\sigma_{AB\bar{B}}(s,r)&=\left[\mathbb{1}_A\oplus S_{B,\bar{B}}(r)\right]\left[\sigma_{AB}^{(\mathrm{M})}(s)\oplus\mathbb{1}_{\bar{B}}\right]\left[\mathbb{1}_A\oplus S_{B,\bar{B}}(r)\right]\\ &=\begin{pmatrix}\sigma_A & \mathcal{E}_{AB} & \mathcal{E}_{A\bar{B}}\\ \mathcal{E}_{AB}^{\mathsf{T}} & \sigma_B & \mathcal{E}_{B\bar{B}}\\ \mathcal{E}_{A\bar{B}}^{\mathsf{T}} & \mathcal{E}_{B\bar{B}}^{\mathsf{T}} & \sigma_{\bar{B}}\end{pmatrix},\end{aligned} \tag{513}$$

where the diagonal elements are

$$\begin{aligned}\sigma_A&=\cosh(2s)\mathbb{1}_2,\\ \sigma_B&=[\cosh(2s)\cosh^2(r)+\sinh^2(r)]\mathbb{1}_2,\\ \sigma_{\bar{B}}&=[\cosh^2(r)+\cosh(2s)\sinh^2(r)]\mathbb{1}_2,\end{aligned} \tag{514}$$

and the non-diagonal elements have the following forms:

$$\begin{aligned}\mathcal{E}_{AB}&=[\cosh(r)\sinh(2s)]Z_2,\\ \mathcal{E}_{B\bar{B}}&=[\cosh^2(s)\sinh(2r)]Z_2,\\ \mathcal{E}_{A\bar{B}}&=[\sinh(2s)\sinh(r)]Z_2.\end{aligned} \tag{515}$$

It was found that for two observers undergoing the finite acceleration, the entanglement vanishes between the lowest-frequency modes. The loss of entanglement is precisely explained as a redistribution of the inertial entanglement into the multipartite quantum correlations among accessible and inaccessible modes from a noninertial perspective. The classical correlations are also lost from the perspective of two accelerated observers but conserved if one of the observers remains inertial.

Leon and Martin-Martinez (2009) investigated the effect of Unruh effect on spin and occupation number entanglement of Dirac fields in the noninertial frame. They analyzed spin Bell states and occupation number entangled state in a relativistic setting, obtained their entanglement dependence on the acceleration. They showed that the acceleration produces a qubit×four-level quantum system state for the spin case, while there is always qubit×qubit for the spinless case despite their apparent similitude. The entanglement degradation in the spin case is greater than in the spinless case. They as well introduced a procedure to consistently erase the spin information from the system and preserving occupation numbers at the same time. Mann and Villalba (2009) studied the speeding-up degradation of entanglement as a function of acceleration for the free scalar field in an accelerated frame.

Moradi (2009) studied the distillability of entanglement of bipartite mixed states of two modes of a free Dirac field in accelerated frames. It was showed that there are certain values of accelerations which will change the state from a distillable one into separable one. Doukas and Hollenberg (2009) studied the loss of spin entanglement for accelerated electrons in electric and magnetic fields by using an open quantum system. They found that the proper time for the extinguishment of entanglement is proportional to the inverse of the acceleration cubed at high Rindler temperature. Ostapchuk and Mann

(2009) studied the generation of entangled fermions by accelerated measurements on the vacuum. (Aspachs et al., 2010) find that the Unruh–Hawking effect acts on a quantum system as a bosonic amplification channel.

Wang and Jing (2010) studied the dynamics of quantum entanglement for Dirac field when the field interacts with noise environment in noninertial frames. They found that the decoherence induced by the noise environment and loss of the entanglement generated by the Unruh effect will influence each other remarkably. In the case of the total system interact with noise environment, the sudden death of entanglement may appear for any acceleration. However, sudden death may only occur when the acceleration parameter is greater than a critical point when only Rob's qubit under decoherence.

Hwang et al. (2011) examined the entanglement of a tripartite of scalar field when one of the three parties is moving with uniform acceleration. The tripartite entanglement exhibits a decreasing behavior but does not completely vanish in the infinite acceleration limit, which is different from the behavior of bipartite entanglement. This fact indicates that the quantum information processing tasks using tripartite entanglement may be possible even if one of the parties approaches to the horizon of the Rindler spacetime.

Wang and Jing (2011) investigated tripartite entanglement of a fermionic system when one or two subsystems accelerated. They found that all the one-tangles decrease with increasing acceleration but never reduce to zero for any acceleration, which is different from the scalar case of scalar field. It was shown that the system has only tripartite entanglement when one or two subsystems with accelerated motion, which means that the acceleration does not effect the entanglement structure of the quantum states. The tripartite entanglement of the case of two observers accelerated decreases much quicker than the one-observer-accelerated case.

Olson and Ralph (2011) studied quantum entanglement between the future region and the past region in the quantum vacuum of the Rindler spacetime. The massless free scalar fields within the future and past light cone was quantized as independent systems. The initial vacuum between the future and past regions became an entangled state of these systems, which exactly mirrors the prepared entanglement between the space-like separated Rindler wedges. This lead to the notion of time-like entanglement. They described a detector which would exhibit thermal response to the vacuum and discussed the feasibility of detecting the Unruh effect.

Wang and Jing (2012) discussed the system–environment dynamics of Dirac fields for amplitude damping and phase damping channels in noninertial systems. They found that the thermal fields generated by the Unruh thermal bath promotes the sudden death of entanglement between the subsystems while postpone the sudden birth of entanglement between the environments. However, no entanglement was generated between the system and environment when the system coupled with the phase damping environment.

Montero and Martín-Martínez (2012) argued that in the infinite acceleration limit, the entanglement in a bipartite system of the fermionic field must be independent of the choice of Unruh modes. Therefore, to compute field entanglement in relativistic quantum information, the tensor product structures should be modified to give rise to physical results.

Khan (2014) studied the dynamics of tripartite entanglement for Dirac fields through linear contraction criterion in the noninertial frames. It is found that the entanglement measurement is not invariant with respect to the partial realignment of different subsystems if one observer is accelerated case, while it is invariant in the two observers accelerated case. It is shown that entanglement is not generated by the acceleration of the frame for any bipartite subsystems. Unlike the bipartite states, the genuine tripartite entanglement does not completely vanish in both one observer accelerated and two observers accelerated cases even in the limit of infinite acceleration.

Dai et al. (2015) discussed the entanglement of two accelerated Unruh–Wald detectors which couple with real scalar fields. The found that the bipartite entanglement of the two qubits suddenly dies when the acceleration of one or more qubits is large enough, which is a result of Unruh thermal bath. Dai et al. (2016) studied the entanglement of three accelerated qubits, each of them is locally coupled with the real scalar field, without causal influence among the qubits or among the fields. They obtained how the entanglement depends on the accelerations of the three qubits. It was found that quantum entanglement would sudden death if the accelerations of two qubits in the tripartite system are large enough.

Metwally (2016) studied the possibility of recovering the entanglement of accelerated qubit and qutrit systems by using weak-reverse measurements. It is found that the accelerated coded local information in the qutrit system is more robust than that encoded in the qubit system. In addition, the non-accelerated information in the qubit system is not affected by the local operation compared with that depicted on qutrit system.

8.1.2. Discordlike correlations

Datta (2009) discussed the QD between two free modes of a scalar field which are observed by two relatively accelerated observers. It was shown that finite amount of QD exists in the regime where there is no distillable entanglement due to the Unruh effect. In addition, they provided evidence for a nonzero amount of QD in the limit of infinite acceleration. Martín-Martínez and Leon (2010) studied the behavior of classical and quantum correlations in a spacetime with an event horizon, comparing fermionic with bosonic fields. They showed the emergence of conservation laws for classical correlations and quantum entanglement, pointing out the crucial role that statistics plays in the entanglement tradeoff across the horizon.

Wang and Deng (2010c) investigated the distribution of classical correlations and QD of Dirac modes among different regions in a noninertial frames. They found that for the Dirac field, the classical correlation decreases with increasing acceleration, which is different from the scalar field case where the classical correlation is independent of the acceleration.

Ramzan (2013) studied the dynamics of GQD and MIN for noninertial observers at finite temperature. It was found that the GQD can be used to distinguish the Bell, Werner, and general type initial quantum states. In addition, sudden transition in the behavior of GQD and MIN depends on the mean photon number of the local environment. In the case of environmental noise is introduced in the system, this effect becomes more prominent. In the case of depolarizing channel, the environmental noise is found to have stronger affect on the dynamics of GQD and MIN as compared to the Unruh effect. Qiang and Zhang (2015) investigated the distribution of GQD among all possible bipartite divisions of a tripartite system for the free Dirac field modes in noninertial frames. As a comparison, they also discussed the geometric measure of entanglement for the same quantum state.

8.1.3. Quantum coherence

Chen et al. (2016c) investigated the behavior of quantum coherence for free scalar and Dirac modes as detected by accelerated observers. They showed that the relative entropy of coherence is destroyed as increasing acceleration of the detectors. In addition, the shared coherence between the accelerated observers vanishes in the infinite acceleration limit for the scalar field, but tends to a non-vanishing value for the Dirac field.

Huang et al. (2016) studied the freezing condition of coherence for accelerated free modes in a relativistic setting beyond the single-mode approximation. They also discussed the behavior of cohering power and decohering power under the Unruh channel. It was found that the quantum coherence can be distributed between different modes, but the coherence lost in the particle mode sector is not transferred entirely to the antiparticle mode sector. They also demonstrated that the robustness of quantum coherence is better than entanglement under the influence of Unruh effect.

Liu et al. (2016c, d) investigated the dynamics of quantum coherence of two-level atoms interacting with the electromagnetic field in the absence and presence of boundaries. They found that for the two-level systems, the quantum coherence cannot be protected from noise without boundaries. However, in the presence of a boundary, the insusceptible of the quantum coherence can be fulfilled when the atoms are close to the boundary and are transversely polarizable. In addition, in the presence of two parallel reflecting boundaries, for some special distances the quantum coherence of atoms can be shielded from the influence of external environment when the atoms have a parallel dipole polarization at arbitrary location between these two boundaries.

8.2. Free field modes beyond the single-mode approximation

Martín-Martínez and León (2009) introduced an arbitrary number of accessible modes when analyzing the Unruh effect on bipartite entanglement degradation. Bruschi et al. (2010) performed that an inertial observer has the freedom to create excitations in any accessible modes Ω_j, $\forall j$ rather than a typical mode. Therefore, one cannot map a single-frequency Minkowski mode into a set of single frequency Rindler modes in an accelerated setting (Bruschi et al., 2010). That is, the single-mode approximation should be relaxed in a general setting. To overcome the shortage of the single-mode approximation, one should employ the Unruh basis which provides an intermediate step between the Minkowski modes and Rindler modes. The relations between the Unruh and the Rindler operators are

$$
\begin{aligned}
C_{\omega,\mathrm{R}} &= \left(\cosh r_\omega\, \hat{a}_{\omega,\mathrm{I}} - \sinh r_\omega\, \hat{b}^\dagger_{\omega,\mathrm{II}}\right), \\
C_{\omega,\mathrm{L}} &= \left(\cosh r_\omega\, \hat{a}_{\omega,\mathrm{II}} - \sinh r_\omega\, \hat{b}^\dagger_{\omega,\mathrm{I}}\right), \\
D^\dagger_{\omega,\mathrm{R}} &= \left(-\sinh r_\omega\, \hat{a}_{\omega,\mathrm{I}} + \cosh r_\omega\, \hat{b}^\dagger_{\omega,\mathrm{II}}\right), \\
D^\dagger_{\omega,\mathrm{L}} &= \left(-\sinh r_\omega\, \hat{a}_{\omega,\mathrm{II}} + \cosh r_\omega\, \hat{b}^\dagger_{\omega,\mathrm{I}}\right),
\end{aligned}
\tag{516}
$$

where $\sinh r_\omega = (e^{2\pi\omega/a} - 1)^{-1/2}$. The Unruh modes are positive-frequency combinations of plane waves in the Minkowski spacetime, but enjoy an important property: they are mapped into single frequency Rindler modes.

For the free scalar fields, the generic Rindler Fock state $|nm, pq\rangle_\omega$ describing both boson and antiboson is

$$
|nm, pq\rangle_\omega := \frac{\hat{a}^{\dagger n}_{\omega,\mathrm{out}}}{\sqrt{n!}} \frac{\hat{b}^{\dagger m}_{\omega,\mathrm{in}}}{\sqrt{m!}} \frac{\hat{b}^{\dagger p}_{\omega,\mathrm{out}}}{\sqrt{p!}} \frac{\hat{a}^{\dagger q}_{\omega,\mathrm{in}}}{\sqrt{q!}} |0\rangle_S, \tag{517}
$$

where the $\pm$ sign is the notation for boson and antiboson, respectively. This allows us to rewrite the Unruh vacuum as (Fabbri and Navarro-Salas, 2005; Bruschi et al., 2012a)

$$
|0_\omega\rangle_\mathrm{U} = \frac{1}{\cosh r_\omega^2} \sum_{n,m=0}^{\infty} \tanh r_\omega^{n+m} |nn, mm\rangle_\omega, \tag{518}
$$

where $|0_\omega\rangle_\mathrm{U}$ is a shortcut notation used to underline that each Unruh mode ω is mapped into a single frequency Rindler mode ω.

One particle Unruh states are defined as $|1_j\rangle_\mathrm{U}^+ = c^\dagger_{\omega,\mathrm{U}}|0\rangle_\mathrm{H}$, $|1_j\rangle_\mathrm{U}^- = d^\dagger_{\omega,\mathrm{U}}|0\rangle_\mathrm{H}$, where $|0\rangle_\mathrm{H}$ denotes the Hartle–Hawking vacuum. The particle and antiparticle creation operators for Unruh modes are defined as

$$
c^\dagger_{\omega,\mathrm{U}} = q_\mathrm{R} C^\dagger_{\omega,\mathrm{R}} + q_\mathrm{L} C^\dagger_{\omega,\mathrm{L}},\ d^\dagger_{\omega,\mathrm{U}} = q_\mathrm{R} D^\dagger_{\omega,\mathrm{R}} + q_\mathrm{L} D^\dagger_{\omega,\mathrm{L}}, \tag{519}
$$

where q_{R} and q_{L} satisfy $|q_{\mathrm{R}}|^2+|q_{\mathrm{L}}|^2=1$. The operator $c^{\dagger}_{\omega,\mathrm{U}}$ in Eq. (519) means the creation of a pair of particles (Bruschi et al., 2012a; Wang et al., 2014b), i.e., a boson with mode ω in the Rindler region I and an antiboson in the Rindler region II. Similarly, the creation operator $d^{\dagger}_{\omega,\mathrm{U}}$ denotes that an antiboson and a boson are created in Rindler region I and II, respectively.

Martín-Martínez and León (2009) introduced an arbitrary number of accessible modes for the Dirac field. They proved that under the single-mode approximation a fermion only has a few accessible levels due to Pauli exclusion principle, which is different from the bosonic fields which has infinite number of excitable levels. This was argued to justify entanglement survival in the fermionic case at the infinite acceleration limit under the single-mode approximation. By relaxing the single-mode approximation, entanglement loss for the Dirac field mode is limited, which comes from fermionic statistics through the characteristic structure. In addition, the surviving entanglement in the infinite acceleration limit is found to be independent of the type of fermionic field and the number of considered accessible modes.

Bruschi et al. (2010) addressed the validity of the single-mode approximation and discussed the behavior of Unruh effect beyond the single-mode approximation. They argued that the single-mode approximation is not valid for arbitrary states in a relativistic setting. In addition, some corrections to previous studies on relativistic quantum information beyond the single-mode approximation are performed both for the bosonic and fermionic cases. They also exhibited a sequence of wave packets where such approximation is justified subject to the peaking constraints which set by some appropriate Fourier transforms.

Bruschi et al. (2012a) analyzed the tradeoff of quantum entanglement between particle and anti-particle modes of a charged bosonic field in a noninertial frame beyond the single-mode approximation. They found that the redistribution of entanglement between bosonic and antibosonic modes does not prevent the entanglement from vanishing in the limit of infinite acceleration. That is, they included antiparticles in the study of bosonic entanglement by analyzing the charged bosonic case and find that mode entanglement always vanishes in this limit. This supports the conjecture that the main differences in the behavior of entanglement in the bosonic field mode and fermionic field mode case are due to the difference between the Bose–Einstein statistics and the Fermi–Dirac statistics.

Brown et al. (2012) demonstrated that quantum correlations measured by the GQD decays to zero in the limit of infinite acceleration, which is in contrast with previous research showing that the degradation of QD vanishes in this limit. They argued that the usable quantum correlations measured by GQD in the large acceleration regime appear severely limited for any protocols. In addition, vanishing of the GQD implies a significant limitation on the usable quantum correlations for large accelerations.

Tian and Jing (2013) studied the MIN for both Dirac and Bosonic fields in non-inertial frames beyond the single-mode approximation. They found that two different behaviors exist between the Dirac and scalar fields: (i) the MIN for Dirac fields persists for any acceleration, while for Bosonic fields this quantity does decay to zero in the limit of infinite acceleration; (ii) the dynamic behaviors of the MIN for scalar fields are quite different from the Dirac fields case in the accelerated frame. In addition, the MIN is found to be more general than the quantum nonlocality related to violation of Bell inequalities.

Richter and Omar (2015) studied the entanglement of Unruh modes shared by two accelerated observers and find some differences in the robustness of entanglement for these states under the effect of Unruh thermal bath. For the initial state prepared in Bell states of free bosonic and fermionic modes, they found that the states $\Psi_{\pm}$ are entangled for any finite accelerations. However, the states $\Phi_{\pm}$ exist entanglement sudden death for some finite accelerations due to the effect of Unruh radiation. They also considered the differences in the behavior of entanglement for fermionic modes and discussed the role that is played by particle statistics. These results suggest that the degradation of entanglement in noninertial frames strongly depends on the occupation patterns of the constituent states.

8.3. Curved spacetime and expanding universe

8.3.1. In the background of a black hole

As discussed in Refs. (Fuentes-Schuller and Mann, 2005; Pan and Jing, 2008b), the role of a Rindler observer in the accelerated frame corresponds to a Schwarzschild observer in the background of a black hole (Fabbri and Navarro-Salas, 2005). In addition, it was found that the effect of Hawking radiation of the black hole on a quantum system can be described by a bosonic amplification channel (Aspachs et al., 2010). In this case, we assume Alice stays stationary at an asymptotically flat region of an external black hole, and Bob is a Schwarzschild observer who hovers near the event horizon of a black hole. The spacetime background near a static and asymptotically flat Schwarzschild black hole, is described by

$$\begin{aligned} ds^2 = &-(1-\frac{2M}{r})dt^2+(1-\frac{2M}{r})^{-1}dr^2 \\ &+r^2(d\theta^2+\sin^2\theta d\varphi^2), \end{aligned} \tag{520}$$

where M represents the mass of the black hole.

Solving the KG equation or Dirac equation near the event horizon of the black hole, one can obtain a set of positive frequency modes propagating in the exterior and interior regions of the event horizon. Here, we introduce the quantization of Dirac fields, in this case the positive (fermions) frequency solutions are found to be

$$\Psi^{+}_{I,\mathbf{k}}=\mathcal{G}e^{-i\omega\mathcal{U}},\ \Psi^{+}_{II,\mathbf{k}}=\mathcal{G}e^{i\omega\mathcal{U}}, \tag{521}$$

where $\mathcal{U} = t - r_*$ and $\mathcal{G}$ is a 4-component Dirac spinor, $\mathbf{k}$ is the wave vector used to label the modes and for massless Dirac field we have $\omega = |\mathbf{k}|$.

In terms of these bases, the Dirac field Ψ can be expanded as

$$\Psi = \int d\mathbf{k}[\hat{a}^{out}_{\mathbf{k}} \Psi^+_{out,\mathbf{k}} + \hat{b}^{out\dagger}_{-\mathbf{k}} \Psi^-_{out,\mathbf{k}} + \hat{a}^{in}_{\mathbf{k}} \Psi^+_{in,\mathbf{k}} + \hat{b}^{in\dagger}_{-\mathbf{k}} \Psi^-_{in,\mathbf{k}}], \tag{522}$$

where $\hat{a}^{out}_{\mathbf{k}}$ and $\hat{b}^{out\dagger}_{\mathbf{k}}$ are the fermionic annihilation and antifermion creation operators acting on the state of the exterior region of the black hole, and $\hat{a}^{in}_{\mathbf{k}}$ and $\hat{b}^{in\dagger}_{\mathbf{k}}$ are the operators acting on the state in the interior region of the black hole. These operators $\hat{a}^{out}_{\mathbf{k}}$ satisfy the canonical anticommutation relations

$$\begin{aligned} &\{\hat{a}^{out}_{\mathbf{k}}, \hat{a}^{out}_{\mathbf{k}'}\} = \delta_{\mathbf{k}\mathbf{k}'}, \\ &\{\hat{a}^{out}_{\mathbf{k}}, \hat{a}^{out\dagger}_{\mathbf{k}'}\} = \{\hat{a}^{out\dagger}_{\mathbf{k}}, \hat{a}^{out\dagger}_{\mathbf{k}'}\} = 0 \end{aligned} \tag{523}$$

where $\{.,.\}$ denotes the anticommutator.

Making analytic continuation for Eq. (521) according to the suggestion of Damour–Ruffini (Damour and Ruffini, 1976), a set of Kruskal modes is obtained. The Kruskal modes can be used to define the Hartle–Hawking vacuum, corresponding to Minkowski vacuum in an inertial frame (Fabbri and Navarro-Salas, 2005). These two sets of operators are related to each other by the Bogoliubov transformation

$$\tilde{a} = \int_{k'} dk' \left[\alpha_{kk'} a_{k'} + \beta^*_{kk'} a^\dagger_{k'}\right], \tag{524}$$

where $\alpha_{kk'}$ and $\beta_{kk'}$ are the Bogoliubov coefficients, which encode information about the spacetime. To quantize the Dirac field beyond the single-mode approximation (Martín-Martínez and León, 2009; Bruschi et al., 2010), we construct a different set of operators in the inside and outside regions of the black hole, which are

$$\begin{aligned} \tilde{c}^\dagger_{\mathbf{k},R} &= \cos r \hat{a}^{out\dagger}_{\mathbf{k}} - \sin r \hat{b}^{in\dagger}_{-\mathbf{k}} \\ \tilde{c}^\dagger_{\mathbf{k},L} &= \cos r \hat{a}^{in\dagger}_{\mathbf{k}} - \sin r \hat{b}^{out\dagger}_{-\mathbf{k}}, \end{aligned} \tag{525}$$

where

$$\cos r = (e^{-8\pi\omega M} + 1)^{-1/2}, \quad \sin r = (e^{8\pi\omega M} + 1)^{-1/2}. \tag{526}$$

A relevant set of annihilation operators can be constructed in an analogous way. These modes with subscripts L and R are left and right Unruh modes. After some calculations, the Hartle–Hawking vacuum is found to be $|0\rangle_H = \bigotimes_{\mathbf{k}} |0_{\mathbf{k}}\rangle_K$, where

$$\begin{aligned} |0_{\mathbf{k}}\rangle_K =& \cos^2 r|0000\rangle - \sin r|0011\rangle \\ &+ \sin r \cos r|1100\rangle - \sin^2 r|1111\rangle. \end{aligned} \tag{527}$$

In the last-written equation

$$|mnm'n'\rangle = |m_{\mathbf{k}}\rangle^+_{out} |n_{-\mathbf{k}}\rangle^-_{in} |m'_{-\mathbf{k}}\rangle^-_{out} |n'_{\mathbf{k}}\rangle^+_{in}, \tag{528}$$

with $\{|n_{-\mathbf{k}}\rangle^-_{in}\}$ and $\{|n_{\mathbf{k}}\rangle^+_{out}\}$ being the orthonormal bases of the inside and outside of the event horizon of the black hole, and the $\{+, -\}$ is used to indicate the fermion and antifermion vacuum states.

Pan and Jing (2008b) discussed the effect of the Hawking temperature of a static and asymptotically flat black hole on the entanglement and teleportation for the free scalar modes. It was demonstrated that the fidelity of teleportation decreases as the Hawking temperature of the black increases, which indicates the thermal bath induced by the Hawking radiation destroys the quantum channel. The final state is absent of any distillable entanglement in the infinite Hawking temperature limit, which corresponds to the case of the black hole evaporating completely.

Ge and Kim (2008) studied the dynamics of entanglement and the fidelity of teleportation in the background of a rotating black hole with extra dimensions. The metric of a d-dimensional black hole is given by

$$\begin{aligned} ds^2 =& -\left[1 - \left(\frac{r_h}{r}\right)^{d-3}\right] dt^2 + \left[1 - \left(\frac{r_h}{r}\right)^{d-3}\right]^{-1} dr^2 \\ &+ r^2 d\Omega^2_{d-2}, \end{aligned} \tag{529}$$

where r_h is the event horizon of the black hole with area $A_d = r_h^{d-2}\Omega_{d-2}$, and Ω_{d-2} is the volume of a unit $(d-2)$-sphere. From Eq. (529), one can obtain the mass of the d-dimensional black hole, which is

$$M = \frac{(d-2) r_h^{d-3} \Omega_{d-2}}{16\pi G_d}, \tag{530}$$

for the d-dimensional Newton's constant G_d. They discussed how the extra dimensions, the black hole's mass and angular momentum parameter, and mode frequency would influence the behavior of quantum entanglement and fidelity in the curved spacetime. They showed that a maximally entangled initial state which is prepared in an inertial frame becomes less entangled in the curved space due to the Hawking radiation. In addition, the degree of entanglement and fidelity of quantum teleportation were found to be degraded with increasing extra dimension parameter and surface gravity of the black hole.

Wang et al. (2009) studied quantum entanglement of the coupled massive scalar field in the spacetime of a Garfinkle–Horowitz–Strominger dilation black hole. The metric for a Garfinkle–Horowitz–Strominger dilation black hole spacetime is (Garfinkle et al., 1991)

$$ds^2 = -\left(\frac{r-2M}{r-2\alpha}\right)dt^2 + \left(\frac{r-2M}{r-2\alpha}\right)^{-1}dr^2 + r(r-2\alpha)d\Omega^2, \tag{531}$$

where M is the mass of the black hole and α is the dilation charge. It was found that entanglement does not depend on the coupling between the scalar field and the gravitational field and the mass of the field. As the dilation parameter α increases, entanglement is destroyed by the Hawking effect. It is interesting to note that in the limit of $\alpha = M$, corresponding to the case of an extreme black hole, the system has no entanglement for any initial state, where its mutual information equals a nonvanishing minimum value.

Wang and Pan (2010a) studied the quantum projective measurements and generation of entangled Dirac particles in the background of a Schwarzschild Black under the single mode approximation. They found that the measurements performed by Bob who locates near the event horizon of the Schwarzschild black hole creates entangled particles. The particles can be detected by Alice who stays stationary at the asymptotically flat region. In addition, the degree of entanglement increases when the Hawking temperature increases. Deng et al. (2010) studied how the Hawking effect of a black hole influences the entanglement distillability of Dirac fields in the Schwarzschild spacetime. It was found that entanglement distillability of the states are influenced both by the Hawking temperature of the black hole and energy of the fields. Although the parameter of the generic entangled states affects the entanglement, it would not change the range in which the states are entangled for the case of generic entangled states.

Martín-Martínez et al. (2010a) analyzed the entanglement degradation provoked by the Hawking effect in a bipartite system near the event horizon of a Schwarzschild black hole beyond the single mode approximation. They determined the degree of entanglement as a function of the frequency of the field modes, the distance of the accelerated observer to the event horizon, and the mass of the black hole. They found that, in the case of Rob is far off the black hole, all the interesting phenomena occur in the vicinity and the presence of event horizons does not effectively degrade the entanglement. They also discussed the localization of Alice and Rob states in the curved spacetime.

Martín-Martínez et al. (2010b) studied the generation of quantum entanglement in the formation of a black hole. It was found that a field in the separable vacuum of a field can evolve to an entangled state under the influence of a dynamical gravitational collapse. They quantified and discussed the origin of this entanglement and found that for micro-black hole formation and the final stages of evaporating black holes, it could even reach the maximal entanglement limit. In addition, fermions are found to be more sensitive than bosons to the quantum entanglement generation, which is helpful in finding experimental evidence of quantum Hawking effect in analog gravity models.

Wang and Pan (2010b) studied how the Hawking radiation influences the redistribution of the entanglement and mutual information in the Schwarzschild spacetime. It was shown that the physically accessible correlations degrade while the unaccessible correlations increase under the Hawking thermal bath. This is partly because the initial correlations prepared in an inertial frame are redistributed between all the bipartite subsystems. In the limit case that the temperature tends to infinity, the accessible mutual information equals just half of its initial value. They also studied the influence of Hawking radiation on the redistribution of the entanglement and mutual information of free Dirac field modes in the Schwarzschild spacetime (Wang and Pan, 2010b). The results showed that the physically accessible correlations degrade while the unaccessible correlations increase with increasing Hawking temperature. That is, the initial quantum entanglement prepared in inertial frame is redistributed between all the bipartite modes due to the influence of Hawking effect. In the limit of infinite Hawking temperature, the physically accessible mutual information equals just half of its initial value. In addition, the unaccessible mutual information between mode A and II equals the mutual information between mode A and I.

Hosler et al. (2012) discussed quantum communication between an observer who free falls into the black hole and an observer hovering over the horizon of a Schwarzschild black hole. It was found that the communication channels degrade due to the effect of the Unruh–Hawking noise. It was shown that for bosonic quantum communication using single-rail and dual-rail encoding, the classical channel capacity reduces to a finite value and the quantum coherent tends to zero by ignoring time dilation which affects all channels equally. That is, quantum coherence is fully removed at infinite acceleration, whereas classical correlation still exists in this limit.

Wang et al. (2014b) studied the dynamics of the discord-type quantum correlation, the measurement-induced disturbance, and classical correlation of Dirac fields in the background of the Garfinkle–Horowitz–Strominger dilation black hole. They showed that all the above mentioned correlations are destroyed as the increase of black hole's dilation charge. Comparing to the inertial systems, the quantum correlation measured by QD is always not symmetric with respect to the measured subsystems, while the measurement-induced disturbance is always symmetric. In addition, the symmetry of QD is found to be influenced by the spacetime curvature produced by the dilation of the black hole.

He et al. (2016) discussed the MIN for Dirac particles in the Garfinkle–Horowitz–Strominger dilation spacetime. They found that as the dilation parameter increases, the physical accessible MIN decreases monotonically. The physical accessible correlation is found to be nonzero when the Hawking temperature is infinite. This is different from the case of scalar fields and owns to the statistical differences between the Fermi–Dirac fields and the Bose–Einstein fields. They also derived the boundary of the MIN related to Bell-violation and found that the former is more general than the Bell nonlocality.

The behavior of monogamy deficit and monogamy asymmetry of quantum steering under the influence of the Hawking effect is studied in (Wang and Jing, 2018a). In the curved spacetime, the monogamy of quantum steering shows an extreme scenario: the first part of a tripartite system cannot individually steer two other parties, while it can steer the collectivity of them. In addition, the monogamy deficit of Gaussian steering is generated due to the influence of the Hawking thermal bath.

8.3.2. In an expanding universe

The spacetime of a homogeneous and isotropic expanding universe is described by the Friedmann–Lemaî tre–Robertson–Walker (FLRW) metric, which is

$$ds^2 = dt^2 - [a(t)]^2(dx^2), \tag{532}$$

for a two-dimensional geometry. By defining the conformal time coordinate η, the FLRW metric equation is rewritten as

$$ds^2 = [a(\eta)]^2(d\eta^2 - dx^2), \tag{533}$$

where $[a(\eta)]^2 = C(\eta)$ is the conformal scale factor. To solve the KG equation in the asymptotic past $\eta \to -\infty$ and the asymptotic future $\eta \to +\infty$ region, we choose the following conformal scale factor

$$C(\eta) = 1 + \epsilon \tanh(\rho\eta)\,, \tag{534}$$

where ϵ and ρ are parameters controlling the total volume and rapidity of the expansion, respectively. In the asymptotic past and future, the FLRW universe is asymptotically flat. The asymptotic solutions of the KG equation in the past and asymptotic future are

$$\begin{aligned} u_k^{\mathrm{in}}(x,\eta) &\quad \underset{\eta\to-\infty}{\longrightarrow} \frac{1}{2\sqrt{\pi\omega_{\mathrm{in}}}} e^{i(kx-\omega_{\mathrm{in}}\eta)}, \\ u_k^{\mathrm{out}}(x,\eta) &\quad \underset{\eta\to\infty}{\longrightarrow} \frac{1}{2\sqrt{\pi\omega_{\mathrm{out}}}} e^{i(kx-\omega_{\mathrm{out}}\eta)}, \end{aligned} \tag{535}$$

where $\omega_{\mathrm{out/in}} = \sqrt{k^2 + m^2(1\pm\epsilon)}$. Considering the properties of hypergeometric functions, the Bogoliubov coefficient matrix of the scalar field in the FLRW spacetime is calculated in diagonal form. After some calculations, the Bogoliubov transform between operators is found to be

$$a_{k,in} = \alpha_k^* a_{k,out} - \beta_k^* a^\dagger_{-k,out}, \tag{536}$$

and

$$a^\dagger_{k,in} = \alpha_k a^\dagger_{k,out} - \beta_k a_{-k,out} \tag{537}$$

where

$$\begin{aligned} \alpha_k &= \sqrt{\frac{\omega_{\mathrm{out}}}{\omega_{\mathrm{in}}}} \frac{\Gamma([1-(i\omega_{\mathrm{in}}/\rho)])\Gamma(-i\omega_{\mathrm{out}}/\rho)}{\Gamma([1-(i\omega_+/\rho)])\Gamma(-i\omega_+/\rho)}, \\ \beta_k &= \sqrt{\frac{\omega_{\mathrm{out}}}{\omega_{\mathrm{in}}}} \frac{\Gamma([1-(i\omega_{\mathrm{in}}/\rho)])\Gamma(i\omega_{\mathrm{out}}/\rho)}{\Gamma([1+(i\omega_-/\rho)])\Gamma(i\omega_-/\rho)}. \end{aligned} \tag{538}$$

To quantize Dirac fields in the FLRW spacetime (Birrell and Davies, 1982; Duncan, 1978; Bergstrom and Goobar, 2006), an appropriate choice for the conformal factor $a(\eta)$ is (Duncan, 1978; Birrell and Davies, 1982)

$$a(\eta) = 1 + \epsilon(1 + \tanh \rho\eta). \tag{539}$$

Similarly, one may obtain the solution of the Dirac fields that behaving as in the asymptotic past and future region of the FLRW spacetime. Then we calculate the relation between the operators in the asymptotic future and past regions and quantize the field. For the Dirac field, the Bogoliubov transformations (Birrell and Davies, 1982) between *in* and *out* modes are

$$\psi_{in}^{(\pm)}(k) = \mathcal{A}_k^{\pm}\psi_{out}^{(\pm)}(k) + \mathcal{B}_k^{\pm}\psi_{out}^{(\mp)*}(k), \tag{540}$$

where the Bogoliubov coefficients $\mathcal{A}_k^{\pm}$ and $\mathcal{B}_k^{\pm}$ that take the form

$$\begin{aligned} \mathcal{A}_k^{\pm} &= \sqrt{\frac{\omega_{out}}{\omega_{\mathrm{in}}}} \frac{\Gamma(1-\frac{i\omega_{in}}{\rho})\Gamma(-\frac{i\omega_{out}}{\rho})}{\Gamma(1-i\zeta^{\mp}_{(+)}/\rho)\Gamma(-i\zeta^{\pm}_{(+)}/\rho)}, \\ \mathcal{B}_k^{\pm} &= \sqrt{\frac{\omega_{out}}{\omega_{in}}} \frac{\Gamma(1-\frac{i\omega_{in}}{\rho})\Gamma(\frac{i\omega_{out}}{\rho})}{\Gamma(1+i\zeta^{\pm}_{(-)}/\rho)\Gamma(i\zeta^{\mp}_{(-)}/\rho)}. \end{aligned} \tag{541}$$

Ball et al. (2006) studied entanglement for scalar field modes in the two-dimensional asymptotically flat Robertson–Walker expanding spacetime. They showed that the expanding universe generates entanglement between modes of the scalar field, which conversely encodes information of the underlying spacetime structure. They calculated the entanglement in the far future, for the scalar field residing in the vacuum state in the distant past. They pointed out how the cosmological parameters of the toy Robertson–Walker spacetime can be extracted from quantum correlations between the field modes.

Ahn and Kim (2007) considered the entanglement of two-mode squeezed states for scalar fields in the Riemannian spacetime. The system is prepared as a two mode squeezed state for continuous variables from an inertial point of view. The initial system is prepared in Unruh mode A and mode B in an inertial frame with the covariance matrix

$$\sigma_{AB}^{(\mathrm{M})}(s)=\begin{pmatrix}\mathcal{A}_i(s) & \mathcal{E}_i(s)\\ \mathcal{E}_i^T(s) & \mathcal{B}_i(s)\end{pmatrix}, \tag{542}$$

where $\mathcal{A}_i(s)=\mathcal{B}_i(s)=\cosh(2s)\mathbb{1}_2$, and $\mathcal{E}_i(s)=\sinh(2s)Z_2$. This setting allows the use of entanglement measure for continuous variables, which can be applied to discuss free and bound entanglement from the point of view from noninertial observer.

Fuentes et al. (2010) found that entanglement was generated between modes of Dirac fields in a two-dimensional Robertson–Walker universe. The entanglement generated by the expansion of the universe is lower than for the bosonic case for some fixed conditions (Ball et al., 2006). It was also found that the entanglement for Dirac fields codifies more information about the underlying spacetime structure than the bosonic case, thereby allowing us to reconstruct more information about the history of the expanding universe. This highlights the importance of the difference between the bosonic and the fermionic statistics to account for relativistic effects on the entanglement of field modes.

Feng et al. (2013) investigated quantum teleportation between the conformal observer Alice and the inertial observer Bob in de Sitter space with both free scalar modes and cavity modes. The fidelity of the teleportation is found to be degraded in both cases, which is due to the Gibbons–Hawking effect associated with the cosmological horizon of the de Sitter space. In both schemes, the cutoff at Planck-scale causes extra modifications to the fidelity of the teleportation comparing with the standard Bunch–Davies choice.

Moradi et al. (2014) studied the spin-particles entanglement between two modes of Dirac field in the expanding Robertson–Walker spacetime. They calculated the Bogoliubov transformations for spin-particles between the asymptotic flat remote past and far future regions. It was showed that the particles–antiparticles entanglement creation when passing from remote past to far future due to the articles creation, while particles entanglement in the remote past degrades into the far future. They derived analytical expressions of logarithmic negativity both for spin particles and for spin-less ones as function of the density of the created particles. In addition, they highlighted the role of spin of particles for the dynamics of entanglement in the Robertson–Walker spacetime.

Feng et al. (2014) studied the quantum correlations and quantum channel of both free scalar and Dirac modes in de Sitter space. They found that the entanglement between the free field modes is degraded due to the radiation associated with the cosmological horizon. They constructed proper Unruh modes admitting general α-vacua beyond the single-mode approximation and found a convergent feature of both the bosonic and fermionic cases. In particular, the convergent points of fermionic entanglement are found to dependent on the choice of α. Moreover, an one-to-one correspondence between the convergent points of entanglement and zero capacity of quantum channels in the de Sitter space was proved.

Wang et al. (2015) studied the parameter estimation for excitations of Dirac fields in the expanding Robertson–Walker universe. The optimal precision of the estimation was found to depend on the dimensionless mass $\tilde{m}$ and dimensionless momentum $\tilde{k}$ of the Dirac particles. The precision of the estimation was obtained by choosing the probe state as an eigenvector of the Hamiltonian. This is because the largest quantum fisher information can be obtained by performing projective measurements implemented by the projectors onto the eigenvectors of specific probe states.

Pierini et al. (2016) investigated the effects of spin on entanglement arising in Dirac field in the Robertson–Walker universe. They present an approach to treat the case which only requires charge conservation, and the case which also requires angular momentum conservation. It was found that in both situations entanglement originated from the vacuum has the same behaviors and does not qualitatively deviates from the spinless case. Differences only arise for the case in which particles or antiparticles are present in the input state.

Liu et al. (2016e) studied the thermodynamical properties of scalar fields in the Robertson–Walker spacetime. They treated scalar fields in the curved spacetime as a quantum system undergoing a non-equilibrium transformation. The out-of-equilibrium features were studied via a formalism which was developed to derive emergent irreversible features and fluctuation relations beyond the linear response regime. They applied these ideas to the expanding universe scenario, therefore the assumptions on the relation between entropy and quantum matter are not required. They provided a fluctuation theorem to understand particle production due to the universe expansion.

8.4. Noninertial cavity modes

Downes et al. (2011) proposed a scheme for storing quantum correlations in the field modes of moving cavities in a flat spacetime. In contrast to previous work where quantum correlations degrade due to the Unruh–Hawking effect, they found that entanglement in such systems is protected. They further discussed the establishment of entanglement and found that the generation of maximally entangled states between the cavities is in principle possible. Like free field modes, the dynamics of the scalar field inside the cavity is also given by the KG equation given in Eq. (503). Under the Dirichlet boundary conditions, solutions of the KG equation are given by the plane waves

$$u_n(t,x) = \frac{1}{\sqrt{n\pi}} \sin\left(\frac{n\pi}{L}[x - x_1]\right) e^{-\frac{in\pi}{L}t}, \tag{543}$$

and the scalar field contained within the cavity walls is

$$\hat{\phi}_A(t,x) = \sum_n (u_n(t,x)\hat{a}_n + u_n^*(t,x)\hat{a}_n^\dagger), \tag{544}$$

where $\hat{a}_n^\dagger$ and $\hat{a}_n$ are the creation and annihilation operators, with $[\hat{a}_n, \hat{a}_{n'}^\dagger] = \delta_{nn'}$. The Dirichlet boundary conditions describe the perfectly reflecting mirrors of the scalar field which is set to vanish on the boundary. Here, Alice's cavity is inertial and Rob's cavity is described by a uniformly accelerating boundary condition.

The world line of Rob's cavity is described by the Rindler coordinates (η, ξ) given in Eq. (507). We assume that Rob is stationary at spatial location $\xi = \xi_1$ for all η, his trajectory in the Minkowski coordinates has the form $x_1(t) = (t^2 + X_1^2)^{1/2}$, where $X_1 = a^{-1}e^{a\xi_1}$, and Rob's proper acceleration is given by $\alpha = X_1^{-1}$. Rob's cavity consists of two mirrors, one at ξ_1 and the other at ξ_2 and stationary with respect to him.

Then, one let Alice and Rob to meet at $t = 0$ with their mirrors aligned, which fixes the position of Alice's cavity as $x_1 = X_1$ and the length of Rob's cavity at $t = 0$ to be $X_2 - X_1 = L$. Therefore, the length of Rob's cavity in Rindler coordinates is $L' = \frac{1}{a}\ln(1 + aL)$ for all t for fixed a. The boundary conditions $\phi[\eta, \xi_1] = \phi[\eta, \xi_2] = 0$ in this case are time-independent since the length L' is a constant. The solutions of the KG equation take the form

$$v_n(\eta,\xi) = \frac{1}{\sqrt{n\pi}} \sin\left(\frac{n\pi}{L'}\xi\right) e^{-\frac{in\pi}{L'}\eta}, \tag{545}$$

where $n \in \{1, 2, \ldots\}$. Therefore, the scalar field inside the cavity is

$$\hat{\phi}_R(\eta,\xi) = \sum_n (v_n(\eta,\xi)\hat{b}_n + v_n^*(\eta,\xi)\hat{b}_n^\dagger), \tag{546}$$

from Rob's perspective, where $\hat{b}_n^\dagger$ and $\hat{b}_n$ are creation and annihilation operators with $[\hat{b}_n, \hat{b}_{n'}^\dagger] = \delta_{nn'}$. The vacuum state is defined by $\hat{b}_n|0\rangle_R = 0$, $\forall n$, where the subscript R indicates Rindler cavity. Assuming the cavity's mirrors are perfectly reflecting. Then one can obtain that if Rob prepares the cavity in a given Rindler state, it will remain in the same state for all times (Avagyan et al., 2002).

Bruschi et al. (2012b) studied whether the nonuniform motion degrades entanglement of a relativistic quantum field that is localized both in space and in time. The field modes in each cavity are discrete and have the frequencies $\omega_n := \sqrt{M^2 + \pi^2 n^2}/\delta$, where $M := \mu\delta$ and the quantum number is $n \in \{1, 2, \ldots\}$. We then assume Rob undergoes accelerated motion. The trajectories of the cavities are given in Fig. 8. They denote U_n as Rob's field modes with positive frequency ω_n before the acceleration and denote $\bar{U}_n$ as Rob's field modes after the acceleration. The two sets of modes are related by the Bogoliubov transformation

$$\bar{U}_m = \sum_n \left(\alpha_{mn}U_n + \beta_{mn}U_n^*\right), \tag{547}$$

where the Bogoliubov coefficient matrices α and β are determined by the motion of the cavity during the acceleration (Birrell and Davies, 1982). Here, the proper acceleration at the center of the cavity is h/δ, where the parameter h should satisfy $h < 2$ to ensure the acceleration at the left end of the cavity is finite. In the region II, the scalar field positive frequency modes with respect to ξ are a discrete set V_n with $n \in \{1, 2, \ldots\}$, and their frequencies at the center of the cavity are $\tilde{\Omega}_n = (\pi hn)/[2\delta\,\mathrm{atanh}(h/2)]$ with respect to the proper time τ.

The Bogoliubov transformation between the two sets of modes can be computed at the junction $t = 0$ (Birrell and Davies, 1982). The coefficient matrices α and β, have small h expansions and have the form

$$\begin{aligned}
\alpha_{nn}^0 &= 1 - \tfrac{1}{240}\pi^2 n^2 h^2 + O(h^4), \\
\alpha_{mn}^0 &= \sqrt{mn}\,\frac{\left(-1 + (-1)^{m-n}\right)}{\pi^2 (m-n)^3} h + O(h^2), \\
\beta_{mn}^0 &= \sqrt{mn}\,\frac{\left(1 - (-1)^{m-n}\right)}{\pi^2 (m+n)^3} h + O(h^2),
\end{aligned} \tag{548}$$

where $m \neq n$. Then one can calculate the state of the cavity modes after the acceleration.

Friis et al. (2012) analyzed quantum entanglement and nonlocality of a massless Dirac field confined to a cavity. The world tube of the cavity consists of inertial and uniformly accelerated segments, and the accelerations are assumed to be small but the travel time is arbitrarily long. The quantum correlations between the field modes in the inertial cavity and the accelerated cavity modes are periodic in the durations of the individual trajectory segments. They found that the loss of quantum correlations can be entirely avoided by tuning the relative durations of the segments. Compared with bosonic

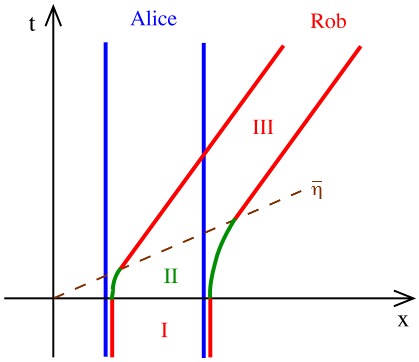

Fig. 8. The figure shows the trajectories of the cavities (Bruschi et al., 2012a): Alice's cavity keeps inertial, Rob's cavity is inertial in region I and is again inertial in region III, Rob's cavity is accelerated in region II. Here, $\bar{\eta}$ is the duration of the acceleration.

correlations, it is easier to calculate the quantum correlations in the fermionic Fock space because the relevant density matrices act in lower dimensional Hilbert space due to the fermionic statistics. Therefore, it is possible to quantify the quantum correlations not only in terms of the entanglement negativity but also in terms of the CHSH inequality.

Brenna et al. (2013) studied the effects of different boundary conditions and coupling forms on the response of a accelerated particle detector in optical cavities. Specifically, they considered cavity fields with periodic, Dirichlet, and Neumann boundary conditions. They demonstrated that the Unruh effect does indeed occur in a cavity, which is independent of the boundary conditions. They found the thermalization properties of the accelerated detector: an accelerated detector evolves to a thermal state whose temperature increases linearly with its acceleration. In a non-perturbative way, it was proven that if the switching process is smooth enough, the detector is thermalized to the Unruh temperature, which is independent of the type of coupling and the boundary conditions.

Bruschi et al. (2013a) discussed how the accelerated motion of a quantum system can be used to generate quantum gates. They present a class of sample travel scenarios in which the nonuniform relativistic motion of a cavity is used to generate two-mode quantum gates in a quantum system with the continuous variables. They found that the degree of entanglement between the cavity modes is produced through resonance of the cavity which appears by repeating periodically trajectory. In addition, they obtained analytical expression of the generated entanglement in terms of the magnitude and direction of the acceleration. The cavity modes are assumed to be initially at rest and the cavity trajectories are constructed through the Bogoliubov transformations. In the covariance matrix formalism, the Bogoliubov transformations are represented by the symplectic matrix $\mathcal{S}$, which has the form

$$s_{kk'} = \begin{pmatrix} \Re(A_{kk'} - B_{kk'}) & \Im(A_{kk'} + B_{kk'}) \\ -\Im(A_{kk'} - B_{kk'}) & \Re(A_{kk'} + B_{kk'}) \end{pmatrix}, \tag{549}$$

where $A_{kk'}$ and $B_{kk'}$ are the Bogoliubov coefficients associated with the trajectory. By assuming $h = aL \ll 1$, the Bogoliubov coefficients can be expanded to the first order in h as

$$A_{kk'} = G_k \delta_{kk'} + A^{(1)}_{kk'}, \; B_{kk'} = B^{(1)}_{kk'}, \tag{550}$$

where $G_k = e^{i\omega_k T}$ are the phases of the state during segments of free evolution, T denotes the total proper time of the segment, and the superscript (1) denotes the first order in h. If the cavity is prepared initially in the vacuum state, the reduced state of the modes after an N-segment trajectory is found to be

$$\sigma_N = (S^N_{kk'})^{Tp} S^N_{kk'}, \tag{551}$$

where

$$S_{kk'} = \begin{pmatrix} s_{kk} & s_{kk'} \\ s_{k'k} & s_{k'k'} \end{pmatrix}, \tag{552}$$

and the transformation $S^N_{kk'}$ corresponds to two mode squeezer that is a two mode entangling gate.

Bruschi et al. (2013b) studied the mode-mixing quantum gates and entanglement in nonuniform accelerated cavities. It was shown that the periodic accelerated motion of the cavity can produce entangling quantum gates between different frequency modes. The resonant condition in the cavities which associates with particle creation is an important feature of the dynamical Casimir effect. It was found that a second resonance, which has attracted less attention because it produces negligible particles, generates a beam splitting quantum gate. This quantum gate leads to a resonant enhancement of quantum entanglement, which can be regarded as the most important evidence of acceleration effects in mechanical oscillators.

Friis et al. (2013a) analyzed relativistic quantum information for quantized scalar, spinor, and photon fields in an accelerated mechanically rigid cavity in the perturbative small acceleration formalism. The scalar field was analyzed with Neumann and Dirichlet boundary conditions, and the photon field was discussed under conductor boundary conditions. The massive Dirac spinor is analyzed with dimensions transverse to the acceleration. It was found that for smooth accelerations, the unitarity of time evolution holds, while for discontinuous accelerations, it fails in 4-dimensions and higher spacetime. The experimental scenario proposed in (Bruschi et al., 2013b) for the scalar field can also apply to the photon field.

Pozas-Kerstjens and Martin-Martinez (2015) analyzed the harvesting of classical and quantum correlations from vacuum for particle detectors. They demonstrated how the spacetime dimensionality, the detectors' physical size, and their internal energy structure would impact the detectors' harvesting ability. They revealed several dependence on these parameters that can optimize the harvesting of quantum entanglement and classical correlations. Furthermore, they found that to harvest vacuum entanglement, smooth switching is more efficient than sudden switching, especially in the case of the detectors that are spacelike separated.

Regula et al. (2016) investigated entanglement generated between the modes of two uniformly accelerated bosonic cavities when interacting with a two-level system. It was found that the inertial and the accelerated cavity become entangled by letting an atom emitting an excitation when it passes through the cavities, but the generated entanglement is degraded against the effects of acceleration. The generated entanglement is affected not only by the accelerated motions of the cavities but also by its transverse dimension which plays the role of an effective mass. In addition, they found that the extra spatial dimensions contribute to the mass of the field. Therefore, if the massless bosonic field is used, the degradation effect of entanglement should not occur.

8.5. Unruh–DeWitt detectors

To model the response of an accelerated detector in a quantum field, the Unruh–DeWitt detector model was performed (Unruh and Wald, 1984). This model consists of a two-level non-interacting atom, which couples to the external scalar field along its world line in a point-like manner (Wald, 1994). The response of the Unruh–DeWitt detector depends on its trajectory and the state of the field. For definiteness and without loss of generality, we consider a uniformly accelerated detector, whose world line is given by Eq. (507).

Here we study two detectors, one named Alice keeps static and the other one named Rob moves with uniform acceleration a for a time duration Δ. Alice's detector is assumed always switched off and Rob's detector is switched on during the time duration Δ. The total Hamiltonian of the system is

$$H_{AR\phi} = H_A + H_R + H_{KG} + H_{\rm int}^{R\phi}, \tag{553}$$

where $H_A = \Omega A^\dagger A$, $H_R = \Omega R^\dagger R$, and Ω is the energy gap of the detectors. The interaction Hamiltonian $H_{\rm int}^{R\phi}(t)$ between Rob's detector and the external scalar field is

$$H_{\rm int}^{R\phi}(t) = \epsilon(t)\int_{\Sigma_t} d^3\mathbf{x}\sqrt{-g}\,\phi(x)[\chi(\mathbf{x})R + \overline{\chi}(\mathbf{x})R^\dagger], \tag{554}$$

where $g \equiv \det(g_{ab})$, and g_{ab} is the Minkowski metric. Moreover,

$$\chi(\mathbf{x}) = (\kappa\sqrt{2\pi})^{-3}\exp(-\mathbf{x}^2/2\kappa^2), \tag{555}$$

is a Gaussian coupling function which vanishes outside a small volume around the detector. This model describes a point-like detector which only interacts with its neighbor fields. The total initial state of detectors-field system has the form

$$|\Psi_{t_0}^{AR\phi}\rangle = |\Psi_{AR}\rangle \otimes |0_M\rangle, \tag{556}$$

where

$$|\Psi_{AR}\rangle = \sin\theta|0_A\rangle|1_R\rangle + \cos\theta|1_A\rangle|0_R\rangle, \tag{557}$$

is the initial state of the detectors, and $|0_M\rangle$ represents that the external scalar field is in vacuum state from an inertial perspective.

In weak coupling case, the final state $|\Psi_{t=t_0+\Delta}^{R\phi}\rangle$ at time $t_0+\Delta$ can be calculated by employing the first order of perturbation over the coupling constant ϵ (Unruh and Wald, 1984). After some calculations, one can find that the final state $|\Psi_t^{R\phi}\rangle$ at time $t = t_0 + \Delta$ is

$$|\Psi_t^{R\phi}\rangle = [I - i(\phi(f)R + \phi(f)^\dagger R^\dagger)]|\Psi_{t_0}^{R\phi}\rangle, \tag{558}$$

where

$$\begin{aligned}\phi(f) &\equiv \int d^4x\sqrt{-g}\,\chi(x)f \\ &= i[a_{RI}(\overline{uE\overline{f}}) - a_{RI}^\dagger(uEf)],\end{aligned} \tag{559}$$

is the operator of the external scalar field (Landulfo and Matsas, 2009; Wald, 1994), and $f \equiv \epsilon(t)e^{-i\Omega t}\chi(\mathbf{x})$ is a compact support complex function. In addition, u is the positive frequency part from a solution of the KG equation in the Rindler metric (Landulfo and Matsas, 2009; Wald, 1994), and E is the difference between the advanced Green function and the retarded Green functions.

Landulfo and Matsas (2009) investigated how the teleportation of a quantum channel is affected by the Unruh effect when one of the entangled detector is accelerated for a finite amount of proper time. They performed a detailed analysis of how the acceleration of the detector and the Unruh effect influence the entangled quantum system. The mutual information and concurrence between the two detectors are calculated and showed that the latter has a sudden death at some fixed finite acceleration. Similarly, the teleportation fidelity exhibits sudden death behavior via the Unruh effect. The values of quantum entanglement and mutual information depend on the time interval along which one of the detectors is accelerated.

Céleri et al. (2010) analyzed the dynamics of QD and classical correlation for a pair of Unruh–DeWitt detectors when one of them is uniformly accelerated, and showed that the discord-type quantum correlation is completely destroyed under the influence of Unruh thermal bath when one detector is in the limit of infinite acceleration, while the classical correlation is nonzero for any acceleration. In particular, unlike the quantum entanglement, the discord-type quantum correlations exhibit sudden-change behavior at certain acceleration parameter. They also discussed how their results can be interpreted when one of the detector hovers near the event horizon of a Schwarzschild black hole.

Ostapchuk et al. (2012) discussed the dynamics of quantum entanglement between a pair of Unruh–DeWitt detectors, one keeps inertial in the flat spacetime, and the other non-uniformly accelerated in some specified way. Each of the detectors coupled to the external scalar quantum field in an indirect way. The primary problem involving nonuniformly accelerated detectors in an event horizon is absent and the Unruh temperature cannot be well defined. By numerical calculation, they demonstrated the quantum entanglement in the weak-coupling limit like those of an oscillator in a bath of time-depending "temperature" proportional to the proper acceleration of the detector, with oscillatory modifications due to non-adiabatic effects.

Different from the Unruh–DeWitt detector model, a localized solution to the problem of entanglement degradation in relativistic settings was performed by Doukas et al. (2013); Dragan et al. (2013a). They prepared a two mode squeezed state between two observers, the inertial Alice and the accelerated Rob. The initial state is

$$\hat{S}_{\rm AB}|0\rangle_{\rm M} = \exp[s(\hat{a}^\dagger\hat{b}^\dagger - \hat{a}\hat{b})]|0\rangle_{\rm M}, \tag{560}$$

where the annihilation operators $\hat{a}$ and $\hat{b}$ are associated with two localized and spatially separated scalar modes $\phi_{\rm A}(x,t)$ and $\phi_{\rm B}(x,t)$, respectively. From the perspective of the accelerated observer, the covariance matrix of the state has the form (Dragan et al., 2013a)

$$\begin{aligned}\sigma =& \mathbb{1} + 2\langle\hat{n}\rangle_U \begin{pmatrix} 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix} + 2\sinh^2 s \begin{pmatrix} |\alpha|^2 & 0 & 0 & 0\\ 0 & |\alpha|^2 & 0 & 0\\ 0 & 0 & |\beta+\beta'^\star|^2 & 2\,\mathrm{Im}(\beta\beta')\\ 0 & 0 & 2\,\mathrm{Im}(\beta\beta') & |\beta-\beta'^\star|^2\end{pmatrix}\\ &+ \sinh 2s \begin{pmatrix} 0 & 0 & -\mathrm{Re}[\alpha(\beta+\beta'^\star)] & -\mathrm{Im}[\alpha(\beta-\beta'^\star)]\\ 0 & 0 & -\mathrm{Im}[\alpha(\beta+\beta'^\star)] & \mathrm{Re}[\alpha(\beta-\beta'^\star)]\\ -\mathrm{Re}[\alpha(\beta+\beta'^\star)] & -\mathrm{Im}[\alpha(\beta+\beta'^\star)] & 0 & 0\\ -\mathrm{Im}[\alpha(\beta-\beta'^\star)] & \mathrm{Re}[\alpha(\beta-\beta'^\star)] & 0 & 0\end{pmatrix},\end{aligned} \tag{561}$$

where

$$\langle\hat{n}\rangle_U = \sum_k \frac{|(\psi_{\rm B}, w_{Ik})|^2}{e^{\frac{2\pi|k|c^2}{a}} - 1}, \quad w_{Ik} = \frac{1}{\sqrt{4\pi|k|c}} e^{i(k\xi - |k|c\tau)}, \tag{562}$$

is the average number of Unruh particles seen by an accelerated detector in the vacuum (Dragan et al., 2013b).

Dragan et al. (2013a) studied the amount of entanglement by using the localized projective detection model and found that the quantum correlations are able to extract from the initial state. It was found that the Unruh thermal noise plays only a minor role in the degradation process of entanglement. The dominant source of degradation is the mismatch between the mode Rob observed in the squeezed state and the mode which is detectable from the accelerated frame. In addition, leakage of initial mode through the Rindler horizon places a limit on Rob's ability to fully measure the state, which leads to an inevitable loss of entanglement that even cannot be corrected by changing the hardware design of the detectors.

Doukas et al. (2013) investigated the quantum entanglement and discord extractable from a two mode squeezed state as considered by two detectors, one inertial and the other accelerated. They found that for large accelerations, the quantum system using localized modes produces qualitatively different properties than that of Unruh modes. Specifically, the quantum entanglement of the given quantum state undergoes a sudden death for some finite acceleration while the discord asymptotes to zero in the infinite acceleration limit.

Tian and Jing (2014) studied the dynamics of freely falling and static two-level detectors interacting with quantized scalar field in de Sitter spacetime. The atomic transition rates is found to depend on both the parameter of de Sitter spacetime and the motion of atoms. They found that the steady states for both cases are always purely thermal states, regardless of the initial states of the detectors. In addition, it was found that the thermal baths will generate entanglement between the freely falling atom and its auxiliary partner. They also calculated the proper time for extinguishment of the entanglement between the detectors.

Lin et al. (2015) studied quantum teleportation modeled by Unruh–DeWitt detectors which initially coupled to a common field. An unknown coherent state of the inertia detector is teleported to the agent Rob with relativistic motion, using a detector pair initially entangled and shared by these two agents. The results showed that average fidelity of the teleportation always drops below the best fidelity value from classical teleportation before the detector pair becomes disentangled. The distortion of the detectors' state can suppress the fidelity significantly even if the detectors are still strongly entangled around the light cone. They pointed out that the dynamics of entanglement is not directly related to the fidelity of quantum teleportation between the detectors observed in Minkowski frame or in quasi-Rindler frame.

Menezes and Svaiter (2016) investigated the radiative processes of entangled and accelerated atoms interacting with an electromagnetic field prepared in the Minkowski vacuum. They discussed the structure of the variation rate of the atomic energy for two atoms moving in different world lines. The contributions of vacuum fluctuations and radiation reaction were identified to the generation of entanglement to the decay of entangled states. The situation where two static atoms are coupled independently to two spatially separated cavities at different temperatures is resembled by the results. In addition, it was found that one of antisymmetric Bell state is a decoherence-free state for equal accelerations.

Wang et al. (2016a) studied how the Unruh thermal noise influences the quantum coherence and compared its behavior with entanglement of the same system. They discussed the frozen condition of coherence and find that the decoherence of detectors' quantum state is irreversible under the effect of Unruh thermal bath without any boundary. Comparing with entanglement which reduces to zero for a finite acceleration, the coherence-type quantum correlation approaches zero only in the limit of an infinite acceleration. They found that the evolution of the detectors' state after the interaction described by the Hamiltonian (553) can also be represented by

$$\rho_t^{AR} = \sum_{\mu\nu} M_\mu^A \otimes M_\nu^R |\Psi_{AR}\rangle\langle\Psi_{AR}| (M_\mu^A \otimes M_\nu^R)^\dagger, \tag{563}$$

where M_μ^A and M_μ^R are the Kraus operators. The Kraus operators act on Rob's state are

$$\begin{aligned} &M_1^R = \begin{pmatrix} \sqrt{1-q} & 0 \\ 0 & \sqrt{1-q} \end{pmatrix}, M_2^R = \begin{pmatrix} 0 & 0 \\ \upsilon\sqrt{q} & 0 \end{pmatrix}, \\ &M_3^R = \begin{pmatrix} 0 & \upsilon \\ 0 & 0 \end{pmatrix}. \end{aligned} \tag{564}$$

Unlike M_μ^R, M_μ^A is an identity matrix because Alice's detector keeps static.

The dynamics of steering between two correlated Unruh–Dewitt detectors when one detector interacts with external scalar field was studied in (Liu and Wang, 2018). The quantum steering is found to be very fragile under the influence of Unruh thermal noise. In addition, the quantum steering experiences "sudden death" for some accelerations, which are quite different from other quantum correlations in the same system.

8.6. Quantum correlations and the dynamical Casimir effect

Like the Unruh effect, dynamical Casimir effect is an important prediction of QFT in relativistic setting (Moore, 1970). The dynamical Casimir effect predicts that relativistic motion of boundary conditions would generate pairs of photons from vacuo. Such prediction has been experimentally observed in a superconducting circuit architecture (Wilson et al., 2011). The modulation of boundary condition, theoretically created by a mirror at relativistic speeds, was achieved by high-frequency modulation of the external magnetic flux threading a superconducting quantum interferometric device (Wilson et al., 2011). The experimental demonstration of the dynamical Casimir effect has triggered a renewed interest in it and has paved the way for the analysis of the role of Casimir radiation as a resource for quantum information tasks (Sabín et al., 2015).

To understand the creation of photons from vacuum fluctuations when the boundaries of the electromagnetic field are modulated, one should quantize the field. In the 2011 experimental observation of this phenomenon, the relativistic moving mirror was simulated by a superconducting quantum interferometric device interrupting a superconducting transmission line (Wilson et al., 2011). The electromagnetic field confined in the transmission line can be described by a flux operator $\Phi(x,t)$, which obeys the KG equation. The solution of the KG equation in the plane-waves basis is (Johansson et al., 2009, 2013):

$$\Phi(x,t) = \sqrt{\frac{\hbar Z_0}{4\pi}} \int \frac{d\omega}{\sqrt{\omega}} (\hat{a}(\omega) e^{i(-\omega t + kx)} + \hat{b}(\omega) e^{i(-\omega t - kx)}), \tag{565}$$

where $k = \omega/\upsilon$, and υ is the speed of light in the transmission line, $Z_0 = \sqrt{L_0/C_0}$ is the characteristic impedance. In Eq. (565), $a(\omega)$ is the annihilation operator of photons that moves into the mirror and $b(\omega)$ denotes the annihilation operator of the photons moving away from the mirror. For sufficiently large superconducting quantum interferometric device plasma frequency (Johansson et al., 2009), the charging energy is small compared with the Josephson energy $E_J(t) = E_J[\Phi(x,t)]$.

Therefore, the superconducting quantum interferometric device can provide a boundary condition to the flux field which is analogous to the boundary condition produced by a relativistic moving mirror. That is,

$$\frac{(2\pi)^2}{\phi_0^2}\Phi(0,t)+\frac{1}{L_0}\left.\frac{\partial\Phi(x,t)}{\partial x}\right|_{x=0}=0, \tag{566}$$

where L_0 is the characteristic inductance per unit length, and the additional term associated with the capacitance is neglected, $\Phi_0 = h/2e$ is the magnetic flux quantum. Inserting Eq. (565) into the boundary condition Eq. (566), one obtains

$$\begin{aligned}&\int_0^\omega \frac{d\omega}{\sqrt{\omega}} ik\frac{L_J(t)}{L_0}(\hat{b}(\omega)-\hat{a}(\omega))e^{-i\omega t}\\ &\quad=\int_0^\omega \frac{d\omega}{\sqrt{\omega}}(\hat{a}(\omega)+\hat{b}(\omega))e^{-i\omega t},\end{aligned} \tag{567}$$

where $L_J(t) = \frac{\phi_0^2/(2\pi)^2}{E_J(t)}$ is the tunable Josephson inductance. Then the pair creation of photons in dynamical Casimir effect can be calculated using scattering theory which describes how the time-dependent boundary condition mixes the input and output modes. By employing the methods discussed in (Johansson et al., 2009, 2013), we obtain the Bogoliubov transformation between the incoming and outgoing modes, which relates the input and the output vacuum state.

Felicetti et al. (2015) discussed how the ultrafast modulation of the qubit-field coupling strength between a superconducting qubit and a single mode of a superconducting resonator mimics the motion of the qubit at relativistic speeds. When the qubit follows an effective oscillatory motion, they find two different regimes. The system is found to experience unbounded photon generation or resemble the anti-JC dynamics, which depends on the oscillation frequency. Moreover, by combining the performed technique with the dynamical Casimir physics, the toolbox for studying relativistic phenomena with superconducting circuits can be enhanced.

Friis et al. (2013b) analyzed the effect of relativistic motion on the fidelity of continuous variable protocol for quantum teleportation and proposed a state-of-the-art technology experiment to test their results. They computed the bounds for the fidelity of teleportation when one of the observers moves with nonuniform acceleration for a finite time, which is degraded due to the observer's motion. The effects of time evolution can be removed by applying time dependent local operations and the effects of acceleration on the fidelity can be isolated in this way. In addition, the origin of the fidelity loss of the quantum teleportation has the same physical regime for particle generation due to motion-underlying the Unruh (or Hawking) radiation or the dynamical Casimir effect.

Alhambra et al. (2014) studied the Casimir–Polder forces experienced by atoms or molecules in optical cavities. They model the quantum systems as qubits, and the electromagnetic field components are modeled as scalar fields with Dirichlet or Neumann boundary conditions. The light–matter interaction model is used to compute the Casimir and Casimir–Polder effects. They found that the diamagnetic term can qualitatively change the Casimir-type forces, or in other words, it can turn a repulsive force into an attractive force and vice versa. To be specific, when this term is present, the atoms are attracted to plates with Dirichlet boundary conditions, while the plate-atom forces are repulsive without this term. They also considered the Neumann boundary condition for the atom with or without diamagnetic coupling term in a cavity, where the forces are found to have opposite sign to that of the Dirichlet cavity. In addition, the microscopic–macroscopic transition was studied in this system, and the results showed that the atoms start to affect the Casimir force similarly to a dielectric medium for increasing number of atoms in the cavity.

Marino et al. (2014) studied the thermal and nonthermal signatures of the Unruh effect in Casimir–Polder forces. They found that the Casimir–Polder forces between two uniformly accelerated atoms exhibit a transition from the short distance thermal-like behavior to a long distance nonthermal behavior. The former is predicted by the Unruh effect and the latter is associated with the breakdown of local inertial descriptions of the system. This effect extends the Unruh thermal response detected by an accelerated observer to the spatially extended system of two particles. They identified the characteristic length scale with the acceleration of the two atoms for this crossover. Their results were derived separating at fourth order in perturbation theory and radiation reaction field to the Casimir–Polder interaction between a pair of atoms separated by a constant distance and linearly coupled to a scalar field.

Sabín and Adesso (2015) investigated the generation of the Einstein–Podolsky–Rosen steering and Gaussian interferometric power under the influence of the dynamical Casimir effect. They computed the quantum steering and the interferometric power generated in the superconducting waveguide interrupted by the superconducting quantum interferometric device. It has been shown that, similar with entanglement and QD (Sabín et al., 2015), the value of the experimental driving amplitude and velocity should be higher than a critical value to overcome the initial thermal noise and to create quantum steering. Conversely, the interferometric power is nonzero for any experimental value of the amplitude and velocity and increases with the increasing average number of thermal photons. In other words, any nonzero squeezing produces interferometric power, while a certain value of squeezing is required to generate quantum steering.

Samos-Sáenz de Buruaga and Sabín (2017) studied how the dynamical Casimir effect influences the behavior of quantum coherence for Gaussian states in continuous variables. They found that quantum coherence is significantly different from zero for any value of the external pump amplitude for the realistic experimental parameters. This means that the Casimir radiation creates quantum coherence for any value of the pump amplitude. In addition, quantum coherence is always greater than QD and entanglement and exhibits a remarkable robustness under the influence of thermal noise. They believe that quantum coherence is a more suitable figure of merit of the quantum character for the dynamical Casimir effect since the experimental requirements for obtaining a dynamical Casimir effect state with finite coherence are less than that of entanglement or QD.

9. Conclusions

Quantum coherence and quantum correlations are fundamental notions of quantum theory. Although the former was defined with respect to a single system, while the latter were for bi- and multipartite systems, they are intimately related to, and can be transformed into each other through an operational way. When considering their characterization and quantification, there are many different methods apply to different situations. They can also be quantified in a similar manner, e.g., by using the (pseudo) geometric distance of two states. This unified framework provides the possibility for understanding the intrinsic connections between these two basic notions.

We concentrated in this work the recent progresses on the above two notions, mainly those discussed from the resource theoretic perspective. After a short introduction, we reviewed in Sections 2 and 3 the various quantifiers of geometric quantum correlations and quantum coherence, which include their definitions and calculations, and some quantitative relations between these measures. As these measures are defined by an optimal (minimum or maximum) distance between the considered state and the set of states without the quantum property one wants to characterize, they can be categorized as the geometric characterization of quantumness.

Building upon the above basic notions, we reviewed in Section 4 the interpretations of the resource theory of quantum coherence. These include the inter-conversions between coherence and quantum correlations such as entanglement and QD established in an operational way, their role on signifying the wave nature of quantum particle, and the various complementarity relations. This section also covers a review of the distillation and formation of quantum coherence for which different free operations and communication schemes are used.

In Section 5, we summarized the recent investigations of typical quantum algorithms from the perspective of quantum coherence, which include the protocol of quantum state merging, the Deutsch–Jozsa algorithm, the Grover search algorithm, the DQC1 algorithm, and quantum metrology such as the PD task. All these show that the various quantitative measure of coherence can provide new viewpoints on the origin of superiority of quantum information processing tasks.

In Section 6, we reviewed the recent progresses on control of quantum correlations and quantum coherence in noisy channels. We first showed that the various quantum correlation and quantum coherence may be frozen for special forms of initial states, and there is universal freezing phenomenon for the distance-based measure of them. We also showed local and nonlocal creation of quantum correlation and quantum coherence, the cohering power of quantum channels, as well as the evolution equation and preservation of quantumness.

In Section 7, we showed some applications of quantum coherence in the related subjects of condensed matter physics. As explicit examples, we summarized role of quantum coherence on studying the long-range order, VBS state, and quantum phase transitions of the many-body systems. These reveal from one side the potential of characterizing quantum coherence from a quantification perspective.

Finally, we showed in Section 8 some progresses for the study of quantum correlations and quantum coherence in relativistic settings. These involve their behaviors for the free field modes with and beyond single-mode approximation, for curved spacetime and expanding universe, for noninertial cavity modes, as well as quantum correlations for particle detectors and the dynamical Casimir effects on these correlations. The dynamics of quantum correlations and quantum coherence under the influence of Unruh temperature, Hawking temperature, expansion rate of the universe, accelerated motion of cavities and detectors, and boundary conditions of the field, has been reviewed. The advantage and disadvantage of free modes and local modes for the implementation of quantum information processing tasks in noninertial frames and curved spacetime are also discussed.

Despite the main progresses summarized above which are of broad interest, there are still many challenging problems need to be solved in the future. We think, some of the valuable research directions may be of the following.

The characterization and quantification of quantum coherence under extended family of free operations. Up to now, most of the proposed coherence measures were based on the axiomatic postulates of Baumgratz et al. (2014), where some of them may be extended in different circumstances. There may exist other coherence measures which are both mathematically rigorous and physically significant, e.g., those under incoherence preserving operations, translationally invariant operations, SIO, and GIO. Moreover, if the postulates (C1–C3) of Baumgratz et al. (2014) are somewhat released, other measures of quantum coherence that are physically relevant may exist. The physical meanings of those coherence measures are worth exploring. We believe that the searching process for various coherence quantifications will deepen our understanding of quantum theory, and new findings can also be expected.

The intrinsic connections between quantum coherence and quantum correlations may be another topic needs to be further considered. Although for relative entropy of coherence, there are some progresses being achieved in the past two years along this line, for most of the other measures the interconversion between coherence and quantum correlations still remain to be exploited. In particular, while the role of quantum correlations (entanglement, discord, *etc.*) in explicit quantum communication and computation tasks has been proved, the role of quantum coherence seems not to be so convinced. The investigation of their relations and their interconversion can thus provide interpretations of quantum coherence from a practical perspective. Moreover, the quantum coherence measures are questioned as they are basis dependent, the establishments of their connections with the basis-independent quantum correlation measures are therefore also significant from a theoretic point of view.

When considering a real physical system, the detrimental effects of environments are unavoidable. Though the related decoherence process have been analyzed extensively via decay of various quantum correlation measures, the quantum coherence measures defined in a rigorous framework provide in a real sense the tool for a quantitative analysis of the decoherence process. Moreover, the robustness of quantum correlations and quantum coherence against the detrimental effects of environment are also different. In general, the former are more fragile under environment coupling than the latter. But if the two can be converted into each other efficiently, one can store the quantum correlation of a bi- or multipartite system by converting it to coherence of a single system, and then convert it back into quantum correlations when being used.

As a resource theory of quantum coherence, what is really the resource aspect of them is in fact seldom considered. As far as we know, the coherence measures have already been related to quantum protocols such as state merging and quantum state discrimination, but we think these are by no means the only two roles they will be played in explicit quantum tasks. Further research, some of the ideas can be borrowed from the study of quantum correlations, may help to reveal their potential role as a physical resource. Moreover, the applications of these coherence measures in other subjects of physics, e.g., whether they can serve as useful order parameters for exploiting novel properties of many-body systems, may be nontrivial topics of future research.

Just as those exciting findings of this field in the past few years, the solution of the (no limited to) above problems, in our opinion, will continue to impact the development of basic fundamental quantum theory and the implementation of various new quantum technologies which strongly depend on quantum correlations and quantum coherence.

Acknowledgments

This work was supported by NSFC (Grant Nos. 11675129, 91536108, 11675052, 11774406, 11504205, and 11774205), National Key R & D Program of China (Grant Nos. 2016YFA0302104, 2016YFA0300600), New Star Project of Science and Technology of Shaanxi Province (Grant No. 2016 KJXX-27), Strategic Priority Research Program of Chinese Academy of Sciences (Grant No. XDB28000000), Hunan Provincial Natural Science Foundation of China (2018JJ1016), and New Star Team of XUPT.

References

Aaronson, B., Franco, R.L., Adesso, G., 2013a. Phys. Rev. A 88, 012120.
Aaronson, B., Franco, R.L., Compagno, G., Adesso, G., 2013b. New J. Phys. 15, 093022.
Aberg, J., 2006. arXiv:quant-ph/0612146.
Adesso, G., Bromley, T.R., Cianciaruso, M., 2016. J. Phys. A 49, 473001.
Adesso, G., Fuentes-Schuller, I., Ericsson, M., 2007. Phys. Rev. A 76, 062112.
Affleck, A., Kennedy, T., Lieb, E.H., Tasaki, H., 1987. Phys. Rev. Lett. 59, 799.
Affleck, A., Kennedy, T., Lieb, E.H., Tasaki, H., 1988. Commun. Math. Phys. 115, 477.
Ahn, D., Kim, M.S., 2007. Phys. Lett. A 366, 202.
Alhambra, A.M., Kempf, A., Martin-Martinez, E., 2014. Phys. Rev. A 89, 033835.
Alsing, P.M., Fuentes-Schuller, I., Mann, R.B., Tessier, T.E., 2006. Phys. Rev. A 74, 032326.
Altman, E., Vosk, R., 2015. Ann. Rev. Condens. Matter Phys. 6, 383.
Amico, L., Fazio, R., Osterloh, A., Vedral, V., 2008. Rev. Mod. Phys. 80, 517.
Amico, L., Rossini, D., Hamma, A., Korepin, V.E., 2012. Phys. Rev. Lett. 108, 240503.
Anderson, P.W., 1958. Phys. Rev. 109, 1492.
Andersson, E., Cresser, J.D., Hall, M.J.W., 2007. J. Mod. Opt. 54, 1695.
Aspachs, M., Adesso, G., Fuentes, I., 2010. Phys. Rev. Lett. 105, 151301.
Atas, Y.Y., Bogomolny, E., Giraud, O., Roux, G., 2013. Phys. Rev. Lett. 110, 084101.
Avagyan, R.M., Saharian, A.A., Yeranyan, A.H., 2002. Phys. Rev. D 66, 085023.
Bagan, E., Bergou, J.A., Cottrell, S.S., Hillery, M., 2016. Phys. Rev. Lett. 116, 160406.
Bai, Y.K., Xu, Y.F., Wang, Z.D., 2014. Phys. Rev. Lett. 113, 100503.
Ball, J.L., Fuentes-Schuller, I., Schuller, F.P., 2006. Phys. Lett. A 359, 550.
Bardarson, J.H., Pollmann, F., Moore, J.E., 2012. Phys. Rev. Lett. 109, 017202.
Bartlett, S.D., Rudolph, T., Spekkens, R.W., 2007. Rev. Mod. Phys. 79, 555.
Basko, D.M., Aleiner, I.L., Altshuler, B.L., 2006. Ann. Phys. 321, 1126.
Baumgratz, T., Cramer, M., Plenio, M.B., 2014. Phys. Rev. Lett. 113, 140401.
Bennett, C.H., Brassard, G., Crépeau, C., Jozsa, R., Peres, A., Wootters, W.K., 1993. Phys. Rev. Lett. 70, 1895.
Bennett, C.H., DiVincenzo, D.P., Shor, P.W., Smolin, J.A., Terhal, B.M., Wootters, W.K., 2001. Phys. Rev. Lett. 87, 077902.
Bennett, C.H., DiVincenzo, D.P., Smolin, J.A., Wootters, W.K., 1996. Phys. Rev. A 54, 3824.
Bera, M.N., Qureshi, T., Siddiqui, M.A., Pati, A.K., 2015. Phys. Rev. A 92, 012118.
Bergstrom, L., Goobar, A., 2006. Cosmology and Particle Astrophysics, second ed. Springer.
Berta, M., Christandl, M., Colbeck, R., Renes, J.M., Renner, R., 2010. Nat. Phys. 6, 659.
Bhatia, R., 1997. Matrix Analysis. Springer-Verlag, New York.
Biham, E., Biham, O., Biron, D., Grassl, M., Lidar, D.A., 1999. Phys. Rev. A 60, 2742.
Birrell, N.D., Davies, P.C.W., 1982. Quantum Fields in Curved Space. Cambridge University Press, Cambridge.
Bohigas, O., Giannoni, M.J., Schmit, C., 1984. Phys. Rev. Lett. 52, 1.
Brandão, F.G.S.L., 2005. Phys. Rev. A 72, 022310.
Brandão, F.G.S.L., Datta, N., 2011. IEEE Trans. Inform. Theory 57, 1754.
Brandão, F.G.S.L., Vianna, R.O., 2006. Int. J. Quantum Inf. 4, 331.
Brenna, W.G., Brown, E.G., Mann, R.B., Martin-Martinez, E., 2013. Phys. Rev. D 88, 064031.

Bromley, T.R., Cianciaruso, M., Adesso, G., 2015. Phys. Rev. Lett. 114, 210401.
Brown, E.G., Cormier, K., Martin-Martinez, E., Mann, R.B., 2012. Phys. Rev. A 86, 032108.
Bruschi, D.E., Dragan, A., Fuentes, I., Louko, J., 2012a. Phys. Rev. A 86, 025026.
Bruschi, D.E., Dragan, A., Lee, A.R., Fuentes, I., Louko, J., 2013a. Phys. Rev. Lett. 111, 090504.
Bruschi, D.E., Dragan, A., Louko, J., Faccio, D., Fuentes, I., 2013b. New J. Phys. 15, 073052.
Bruschi, D.E., Fuentes, I., Louko, J., 2012b. Phys. Rev. D 85, 061701.
Bruschi, D.E., Louko, J., Martin-Martinez, E., Dragan, A., Fuentes, I., 2010. Phys. Rev. A 82, 042332.
Bu, K., Anand, N., Singh, U., 2018. Phys. Rev. A 97, 032342.
Bu, K., Kumar, A., Zhang, L., Wu, J., 2017a. Phys. Lett. A 381, 1670.
Bu, K., Singh, U., Fei, S.M., Pati, A.K., Wu, J., 2017b. Phys. Rev. Lett. 119, 150405.
Bu, K., Singh, U., Zhang, L., Wu, J., 2016a. arXiv:1603.06715.
Bu, K., Swati, , Singh, U., Wu, J., 2016b. Phys. Rev. A 94, 052335.
Buscemi, F., Datta, N., 2010. IEEE Trans. Inform. Theory 56, 1447.
Cakmak, B., Karpat, G., Fanchini, F.F., 2015. Entropy 17, 790.
Céleri, L.C., Landulfo, A.G.S., André, R.M., Matsas, G.E.A., 2010. Phys. Rev. A 81, 062130.
Chang, L., Luo, S., 2013. Phys. Rev. A 87, 062303.
Chen, B., Fei, S.M., 2018. Quantum Inf. Process. 17, 107.
Chen, J.J., Cui, J., Zhang, Y.R., Fan, H., 2016a. Phys. Rev. A 94, 022112.
Chen, J., Grogan, S., Johnston, N., Li, C.K., Plosker, S., 2016b. Phys. Rev. A 94, 042313.
Chen, L., Aulbach, M., Hajdušek, M., 2014. Phys. Rev. A 89, 042305.
Chen, L., Chitambar, E., Modi, K., Vacanti, G., 2011. Phys. Rev. A 83, 020101.
Chen, X., Wu, C.F., Su, H.Y., Ren, C.L., Chen, J.L., 2016c. arXiv:1601.02741.
Cheng, C.C., Wang, Y., Guo, J.L., 2016. Ann. Phys. 374, 237.
Cheng, S., Hall, M.J.W., 2016. Phys. Rev. A 92, 042101.
Cheng, W.W., Li, J.X., Shan, C.J., Gong, L.Y., Zhao, S.M., 2015. Quantum Inf. Process. 14, 2535.
Cheng, W.W., Wang, X.Y., Sheng, Y.B., Gong, L.Y., Zhao, S.M., Liu, J.M., 2017. Sci. Rep. 7, 42360.
Chin, S., 2017. Phys. Rev. A 96, 042336.
Chitambar, E., Gour, G., 2016a. Phys. Rev. Lett. 117, 030401.
Chitambar, E., Gour, G., 2016b. Phys. Rev. A 94, 052336.
Chitambar, E., Hsieh, M.-H., 2016. Phys. Rev. Lett. 117, 020402.
Chitambar, E., Streltsov, A., Rana, S., Bera, M.N., Adesso, G., Lewenstein, M., 2016. Phys. Rev. Lett. 116, 070402.
Cianciaruso, M., Bromley, T.R., Roga, W., Franco, R.L., Adesso, G., 2015. Sci. Rep. 5, 10177.
Ciccarello, F., Giovannetti, V., 2012. Phys. Rev. A 85, 010102.
Ciccarello, F., Tufarelli, T., Giovannetti, V., 2014. New J. Phys. 16, 013038.
Coles, P.J., Berta, M., Tomamichel, M., Wehner, S., 2015. Rev. Mod. Phys. 89, 015002.
Costa, A.C.S., Angelo, R.M., 2013. Phys. Rev. A 87, 032109.
Crispino, L.C.B., Higuchi, A., Matsas, G.E.A., 2008. Rev. Mod. Phys. 80, 787.
Dai, Y., Shen, Z., Shi, Y., 2015. J. High Energy Phys. 09, 071.
Dai, Y., You, W., Dong, Y., Zhang, C., 2017. Phys. Rev. A 96, 062308.
Dai, Y., Shen, Z., Shi, Y., 2016. Phys. Rev. D 94, 025012.
Dakić, B., Lipp, Y.O., Ma, X., Ringbauer, M., Kropatschek, S., Barz, S., Paterek, T., Vedral, V., Zeilinger, A., Brukner, Č., Walther, P., 2012. Nat. Phys. 8, 666.
Dakić, B., Vedral, V., Brukner, Č., 2010. Phys. Rev. Lett. 105, 190502.
Damour, T., Ruffini, R., 1976. Phys. Rev. D 14, 332.
Datta, A., 2009. Phys. Rev. A 80, 052304.
Datta, A., Gharibian, S., 2009. Phys. Rev. A 79, 042325.
Datta, A., Shaji, A., Caves, C.M., 2008. Phys. Rev. Lett. 100, 050502.
Datta, N., 2009b. IEEE Trans. Inform. Theory 55, 2816.
Datta, N., 2009c. Int. J. Quantum Inf. 7, 475.
Datta, N., Dorlas, T., Jozsa, R., Benatti, F., 2014. J. Math. Phys. 55, 062203.
Debarba, T., Maciel, T.O., Vianna, R.O., 2012. Phys. Rev. A 86, 024302.
Deng, J., Wang, J., Jing, J., 2010. Phys. Lett. B 695, 495.
Deutsch, D., Jozsa, R., 1992. Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci. 439, 553.
Deutsch, J.M., 1991. Phys. Rev. A 43, 2046.
Dillenschneider, R., 2008. Phys. Rev. B 78, 224413.
Doukas, J., Brown, E.G., Dragan, A., Mann, R.B., 2013. Phys. Rev. A 87, 012306.
Doukas, J., Hollenberg, L., 2009. Phys. Rev. A 79, 052109.
Downes, T.G., Fuentes, I., Ralph, T.C., 2011. Phys. Rev. Lett. 106, 210502.
Dragan, A., Doukas, J., Martin-Martinez, E., 2013a. Phys. Rev. A 87, 052326.
Dragan, A., Doukas, J., Martin-Martinez, E., Bruschi, D.E., 2013b. Classical Quantum Gravity 30, 235006.
Du, S., Bai, Z., 2015. Ann. Phys. 359, 136.
Du, S., Bai, Z., Guo, Y., 2015. Phys. Rev. A 91, 052120.
Duncan, A., 1978. Phys. Rev. D 17, 964.
Eisert, J., Cramer, M., Plenio, M.B., 2010. Rev. Mod. Phys. 82, 277.
Eisert, J., Friesdorf, M., Gogolin, C., 2015. Nat. Phys. 11, 124.
Englert, B.-G., 1996. Phys. Rev. Lett. 77, 2154.
Essler, F.H.L., Korepin, V.E., Schoutens, K., 1992. Phys. Rev. Lett. 68, 2960.
Fabbri, A., Navarro-Salas, J., 2005. Modeling Black Hole Evaporation. Imperial College Press, London.
Fan, H., Korepin, V., Roychowdhury, V., 2004. Phys. Rev. Lett. 93, 227203.
Fan, H., Lloyd, S., 2005. J. Phys. A 38, 5285.
Farías, O.J., Latune, C.L., Walborn, S.P., Davidovich, L., Ribeiro, P.H.S., 2009. Science 324, 1414.
Felicetti, S., Sabin, C., Fuentes, I., Lamata, L., Romero, G., Solano, E., 2015. Phys. Rev. B 92, 064501.
Feng, J., Yang, W.L., Zhang, Y.Z., Fan, H., 2013. Phys. Lett. B 719, 430.
Feng, J., Zhang, Y.Z., Gould, M.D., Fan, H., Sun, C.Y., Yang, W.L., 2014. Ann. Phys. 351, 872.
Ficek, Z., Swain, S., 2005. Quantum Interference and Coherence: Theory and Experiments. In: Springer Series in Optical Sciences.

Fidkowski, L., 2010. Phys. Rev. Lett. 104, 130502.
Filho, J.L.C.C., Saguia, A., Santos, L.F., Sarandy, M.S., 2017. Phys. Rev. B 96, 014204.
Flammia, S.T., Hamma, A., Hughes, T.L., Wen, X.G., 2009. Phys. Rev. Lett. 103, 261601.
Friis, N., Lee, A.R., Bruschi, D.E., Louko, J., 2012. Phys. Rev. D 85, 025012.
Friis, N., Lee, A.R., Louko, J., 2013a. Phys. Rev. D 88, 064028.
Friis, N., Lee, A.R., Truong, K., Sabín, C., Solano, E., Johansson, G., Fuentes, I., 2013b. Phys. Rev. Lett. 110, 113602.
Fuentes, I., Mann, R.B., Martín-Martínez, E., Moradi, S., 2010. Phys. Rev. D 82, 045030.
Fuentes-Schuller, I., Mann, R.B., 2005. Phys. Rev. Lett. 95, 120404.
Galve, F., Giorgi, G.L., Zambrini, R., 2011. Europhys. Lett. 96, 40005.
Galve, F., Plastina, F., Paris, M.G.A., Zambrini, R., 2013. Phys. Rev. Lett. 110, 010501.
Garfinkle, D., Horowitz, G.T., Strominger, A., 1991. Phys. Rev. D 43, 3140.
Ge, X., Kim, S.P., 2008. Classical Quantum Gravity 25, 075011.
Genovese, M., 2005. Phys. Rep. 413, 319.
Geraedts, S.D., Nandkishore, R., Regnault, N., 2016. Phys. Rev. B 93, 174202.
Giorgi, G.L., 2013. Phys. Rev. A 88, 022315.
Giovannetti, V., Lloyd, S., Maccone, L., 2011. Nature Photon. 5, 222.
Girolami, D., 2014. Phys. Rev. Lett. 113, 170401.
Girolami, D., Adesso, G., 2011. Phys. Rev. A 84, 052110.
Girolami, D., Paternostro, M., Adesso, G., 2011. J. Phys. A 44, 352002.
Girolami, D., Tufarelli, T., Adesso, G., 2013. Phys. Rev. Lett. 110, 240402.
Goold, J., Gogolin, C., Clark, S.R., Eisert, J., Scardicchio, A., Silva, A., 2015. Phys. Rev. B 92, 180202.
Gornyi, I.V., Mirlin, A.D., Polyakov, D.G., 2005. Phys. Rev. Lett. 95, 206603.
Gour, G., 2005. Phys. Rev. A 71, 012318.
Gour, G., Müller, M.P., Narasimhachar, V., Spekkens, R.W., Halpern, N.Y., 2015. Phys. Rep. 583, 1.
Grover, L.K., 1997. Phys. Rev. Lett. 79, 325.
Gühne, G., Tóth, G., 2009. Phys. Rep. 474, 1.
Guo, Y., 2013. Int. J. Mod. Phys. B 27, 1350067.
Guo, Y., 2016. Sci. Rep. 6, 25241.
Guo, Y., Hou, J.C., 2013a. J. Phys. A 46, 155301.
Guo, Y., Hou, J., 2013b. J. Phys. A 46, 325301.
Guo, Y., Li, X., Li, B., Fan, H., 2015. Int. J. Theor. Phys. 54, 2022.
Haikka, P., Johnson, T.H., Maniscalco, S., 2013. Phys. Rev. A 87, 010103.
Haldane, F.D.M., 1983a. Phys. Lett. A 93, 464.
Haldane, F.D.M., 1983b. Phys. Rev. Lett. 50, 1153.
Hamieh, S., Kobes, R., Zaraket, H., 2004. Phys. Rev. A 70, 052325.
Hamma, A., Ionicioiu, R., Zanardi, P., 2005. Phys. Rev. A 71, 022315.
Hassan, A.S.M., Joag, P.S., 2013. Europhys. Lett. 103, 10004.
Hassan, A.S.M., Lari, B., Joag, P.S., 2012. Phys. Rev. A 85, 024302.
Hawking, S.W., 1976. Phys. Rev. D 14, 2460.
Hayden, P., Leung, D.W., Winter, A., 2006. Commun. Math. Phys. 265, 95.
He, J., Xu, S., Ye, L., 2016. Phys. Lett. B 756, 278.
Henderson, L., Vedral, V., 2001. J. Phys. A 34, 6899.
Hillery, M., 2016. Phys. Rev. A 93, 012111.
Horodecki, R., Horodecki, M., Horodecki, P., 1996. Phys. Lett. A 222, 21.
Horodecki, M., Horodecki, P., Horodecki, R., Oppenheim, J., Sen, A., Sen, U., Synak-Radtke, B., 2005a. Phys. Rev. A 71, 062307.
Horodecki, M., Horodecki, P., Horodecki, R., Oppenheim, J., Sen(De), A., Sen, U., Synak, B., 2005b. Phys. Rev. A 71, 062307.
Horodecki, M., Horodecki, P., Oppenheim, J., 2003a. Phys. Rev. A 67, 062104.
Horodecki, M., Oppenheim, J., Winter, A., 2005c. Nature 436, 673.
Horodecki, M., Shor, P.W., Ruskai, M.B., 2003b. Rev. Mod. Phys. 15, 629.
Horodecki, R., Horodecki, P., Horodecki, M., Horodecki, K., 2009. Rev. Mod. Phys. 81, 865.
Hosler, D., van de Bruck, C., Kok, P., 2012. Phys. Rev. A 85, 042312.
Hu, M.L., 2011. Phys. Lett. A 375, 2140.
Hu, X., 2016. Phys. Rev. A 94, 012326.
Hu, M.L., Fan, H., 2012a. Ann. Phys. 327, 2343.
Hu, M.L., Fan, H., 2012b. Ann. Phys. 327, 851.
Hu, M.L., Fan, H., 2015a. New J. Phys. 17, 033004.
Hu, M.L., Fan, H., 2015b. Phys. Rev. A 91, 052311.
Hu, X., Fan, H., 2015c. Phys. Rev. A 91, 022301.
Hu, M.L., Fan, H., 2016a. Sci. Rep. 6, 29260.
Hu, X., Fan, H., 2016b. Sci. Rep. 6, 34380.
Hu, M.L., Fan, H., 2017. Phys. Rev. A 95, 052106.
Hu, M.L., Fan, H., 2018. Phys. Rev. A 98, 022312.
Hu, M.L., Lian, H.L., 2015. Ann. Phys. 362, 795.
Hu, M.L., Shen, S.Q., Fan, H., 2017. Phys. Rev. A 96, 052309.
Hu, M.L., Sun, J., 2015. Ann. Phys. 354, 265.
Hu, M.L., Tian, D.P., 2014. Ann. Phys. 343, 132.
Hu, M.L., Wang, X.M., Fan, H., 2018. arXiv:1802.03540.
Hu, X., Fan, H., Zhou, D.L., Liu, W.M., 2012. Phys. Rev. A 85, 032102.
Hu, X., Fan, H., Zhou, D.L., Liu, W.M., 2013a. Phys. Rev. A 87, 032340.
Hu, X., Fan, H., Zhou, D.L., Liu, W.M., 2013b. Phys. Rev. A 88, 012315.
Hu, X., Gu, Y., Gong, Q., Guo, G., 2011. Phys. Rev. A 84, 022113.
Hu, X., Milne, A., Zhang, B., Fan, H., 2015. Sci. Rep. 6, 19365.
Huang, Z., Situ, H., Zhang, C., 2016. arXiv:1609.02295.
Huang, Z., Situ, H., Zhang, C., 2017. Int. J. Theor. Phys. 56, 2178.

Huber, M., Perarnau-Llobet, M., Hovhannisyan, K.V., Skrzypczyk, P., Klöckl, C., Brunner, N., Acín, A., 2015. New J. Phys. 17, 065008.
Hwang, M.-R., Park, D., Jung, E., 2011. Phys. Rev. A 83, 012111.
Ishizaka, S., Hiroshima, T., 2000. Phys. Rev. A 62, 022310.
Ivanovic, I., 1981. J. Phys. A 14, 3241.
Jakóbczyk, L., Frydryszak, A., Ługiewicz, P., 2016. Phys. Lett. A 380, 1535.
Jaynes, E.T., 1957a. Phys. Rev. 106, 620.
Jaynes, E.T., 1957b. Phys. Rev. 108, 171.
Jevtic, S., Pusey, M., Jennings, D., Rudolph, T., 2014. Phys. Rev. Lett. 113, 020402.
Johansson, J.R., Johansson, G., Wilson, C.M., Delsing, P., Nori, F., 2013. Phys. Rev. A 87, 043804.
Johansson, J.R., Johansson, G., Wilson, C.M., Nori, F., 2009. Phys. Rev. Lett. 103, 147003.
Jozsa, R., Linden, N., 2003. Proc. R. Soc. A 459, 2011.
Karpat, G., Çakmak, B., Fanchini, F.F., 2014. Phys. Rev. B 90, 104431.
Karpat, G., Gedik, Z., 2011. Phys. Lett. A. 375, 4166.
Keser, A.C., Ganeshan, S., Refael, G., Galitski, V., 2016. Phys. Rev. B 94, 085120.
Khan, S., 2014. Ann. Phys. 348, 270.
Kitaev, A., Preskill, J., 2006. Phys. Rev. Lett. 96, 110404.
Knill, E., Laflamme, R., 1998. Phys. Rev. Lett. 81, 5672.
Konrad, T., deMelo, F., Tiersch, M., Kasztelan, C., Aragão, A., Buchleitner, A., 2008. Nat. Phys. 4, 99.
Korzekwa, K., Lostaglio, M., Jennings, D., Rudolph, T., 2014. Phys. Rev. A 89, 042122.
Laflamme, R., Cory, D.G., Negrevergne, C., Viola, L., 2002. Quantum Inf. Comput. 2, 166.
Lambert, N., Chen, Y.-N., Cheng, Y.-C., Li, C.-M., Chen, G.-Y., Nori, F., 2013. Nat. Phys. 9, 10.
Landulfo, A.G.S., Matsas, G.E.A., 2009. Phys. Rev. A 80, 032315.
Lei, S., Tong, P., 2016. Quantum Inf. Process. 15, 1811.
Leon, J., Martin-Martinez, E., 2009. Phys. Rev. A 80, 012314.
Levi, E., Heyl, M., Lesanovsky, I., Garrahan, J.P., 2016. Phys. Rev. Lett. 116, 237203.
Levin, M., Wen, X.G., 2006. Phys. Rev. Lett. 96, 110405.
Lewenstein, M., Sanpera, A., 1998. Phys. Rev. Lett. 80, 2261.
Li, H., Haldane, F.D., 2008. Phys. Rev. Lett. 101, 4962.
Li, L., Wang, Q.W., Shen, S.Q., Li, M., 2016. Europhys. Lett. 114, 10007.
Li, Y.C., Lin, H.Q., 2016. Sci. Rep. 6, 26365.
Li, Z., Wang, X.M., Zhou, W., Hu, M.L., 2017. Sci. Rep. 7, 3342.
Lin, S.Y., Chou, C.H., Hu, B.L., 2015. Phys. Rev. D 91, 084063.
Ling, Y., He, S., Qiu, W., Zhang, H., 2007. J. Phys. A 40, 9025.
Liu, C.L., Guo, Y.Q., Tong, D.M., 2017a. Phys. Rev. A 96, 062325.
Liu, C.L., Yu, X.D., Xu, G.F., Tong, D.M., 2016a. Quantum Inf. Process. 15, 4189.
Liu, C.L., Zhang, D.J., Yu, X.D., Ding, Q.M., Liu, L.J., 2017b. Quantum Inf. Process. 16, 198.
Liu, F., Li, F., Chen, J., Xing, W., 2016b. Quantum Inf. Process. 15, 3459.
Liu, N., Goold, J., Fuentes, I., Vedral, V., Modi, K., Bruschi, D.E., 2016e. Classical Quantum Gravity 33, 035003.
Liu, S., Mu, L.Z., 2015a. Phys. Rev. A 91, 042133.
Liu, T.H., Wang, J., 2018. Ann. Phys. 390, 334.
Liu, X.B., Tian, Z.H., Wang, J.C., Jing, J.L., 2016c. Ann. Phys. 366, 102.
Liu, X.B., Tian, Z.H., Wang, J.C., Jing, J.L., 2016d. Quantum Inf. Process. 15, 3677.
Lostaglio, M., Jennings, D., Rudolph, T., 2015a. Nat. Commun. 6, 6383.
Lostaglio, M., Korzekwa, K., Jennings, D., Rudolph, T., 2015b. Phys. Rev. X 5, 021001.
Luitz, D.J., Laflorencie, N., Alet, F., 2015. Phys. Rev. B 91, 081103.
Luo, S.L., 2003. Phys. Rev. Lett. 91, 180403.
Luo, S.L., 2006. Phys. Rev. A 73, 022324.
Luo, S., 2008. Phys. Rev. A 77, 022301.
Luo, S., Fu, S., 2010a. Phys. Rev. A 82, 034302.
Luo, S., Fu, S., 2010b. Europhys. Lett. 92, 20004.
Luo, S., Fu, S., 2011. Phys. Rev. Lett. 106, 120401.
Luo, S., Fu, S., Oh, C.H., 2012. Phys. Rev. A 85, 032117.
Ma, F.W., Liu, S.X., Kong, X.M., 2011. Phys. Rev. A 84, 042302.
Ma, J., Yadin, B., Girolami, D., Vedral, V., Gu, M., 2016a. Phys. Rev. Lett. 116, 160407.
Ma, T., Zhao, M.J., Fei, S.M., Long, G.L., 2016b. Phys. Rev. A 94, 042312.
Ma, T., Zhao, M.J., Wang, Y.K., Fei, S.M., 2014. Sci. Rep. 4, 6336.
Ma, T., Zhao, M.J., Zhang, H.J., Fei, S.M., Long, G.L., 2017. Phys. Rev. A 95, 042328.
Malvezzi, A.L., Karpat, G., Çakmak, B., Fanchini, F.F., Debarba, T., Vianna, R.O., 2016. Phys. Rev. B 93, 184428.
Mani, A., Karimipour, V., 2015. Phys. Rev. A 92, 032331.
Mann, R.B., Villalba, V.M., 2009. Phys. Rev. A 80, 022305.
Marino, J., Noto, A., Passante, R., 2014. Phys. Rev. Lett. 113, 020403.
Martín-Martínez, E., Garay, L.J., León, J., 2010a. Phys. Rev. D 82, 064006.
Martín-Martínez, E., Garay, L.J., León, J., 2010b. Phys. Rev. D 82, 064028.
Martín-Martínez, E., León, J., 2009. Phys. Rev. A 80, 042318.
Martín-Martínez, E., Leon, J., 2010. Phys. Rev. A 81, 032320.
Marvian, I., 2012. Symmetry, Asymmetry and Quantum Information. (Ph.D. thesis), University of Waterloo, https://uwspace.uwaterloo.ca/handle/10012/7088.
Marvian, I., Spekkens, R.W., 2014. Nat. Commun. 5, 3821.
Marvian, I., Spekkens, R.W., 2016. Phys. Rev. A 94, 052324.
Marvian, I., Spekkens, R.W., Zanardi, P., 2016. Phys. Rev. A 93, 052331.
Menezes, G., Svaiter, N.F., 2016. Phys. Rev. A 93, 052117.
Metwally, N., 2016. Europhys. Lett. 116, 60006.
Mirafzali, S.Y., Sargolzahi, I., Ahanj, A., Javidan, K., Sarbishaei, M., 2013. Quantum Inf. Comput. 13, 479.
Misra, A., Singh, U., Bhattacharya, S., Pati, A.K., 2016. Phys. Rev. A 93, 052335.

Miszczak, J.A., Puchała, Z., Horodecki, P., Uhlmann, A., Zyczkowski, K., 2009. Quantum Inf. Comput. 9, 103.
Modak, R., Mukerjee, S., 2015. Phys. Rev. Lett. 115, 230401.
Modi, K., Brodutch, A., Cable, H., Paterek, T., Vedral, V., 2012. Rev. Mod. Phys. 84, 1655.
Modi, K., Paterek, T., Son, W., Vedral, V., Williamson, M., 2010. Phys. Rev. Lett. 104, 080501.
Mondal, D., Pramanik, T., Pati, A.K., 2017. Phys. Rev. A 95, 010301.
Montealegre, J.D., Paula, F.M., Saguia, A., Sarandy, M.S., 2013. Phys. Rev. A 87, 042115.
Montero, M., Martín-Martínez, E., 2012. Phys. Rev. A 85, 024301.
Moore, G.T., 1970. J. Math. Phys. 11, 2679.
Moradi, S., 2009. Phys. Rev. A 79, 064301.
Moradi, S., Pierini, R., Mancini, S., 2014. Phys. Rev. D 89, 024022.
Nakano, T., Piani, M., Adesso, G., 2013. Phys. Rev. A 88, 012117.
Nandkishore, R., Huse, D.A., 2015a. Ann. Rev. Condens. Matter Phys. 6, 15.
Nandkishore, R., Huse, D.A., 2015b. Annu. Rev. Condens. Matter Phys. 6, 15.
Napoli, C., Bromley, T.R., Cianciaruso, M., Piani, M., Johnston, N., Adesso, G., 2016. Phys. Rev. Lett. 116, 150502.
Narasimhachar, V., Gour, G., 2015. Nat. Commun. 6, 7689.
Nielsen, M.A., Chuang, I.L., 2000. Quantum Computation and Quantum Information. Cambridge University Press, Cambridge.
Oganesyan, V., Huse, D.A., 2007. Phys. Rev. B 75, 155111.
Okrasa, M., Walczak, Z., 2012. Europhys. Lett. 98, 40003.
Ollivier, H., Zurek, W.H., 2001. Phys. Rev. Lett. 88, 017901.
Olson, S.J., Ralph, T.C., 2011. Phys. Rev. Lett. 106, 110404.
Oppenheim, J., Horodecki, M., Horodecki, P., Horodecki, R., 2002. Phys. Rev. Lett. 89, 180402.
Osborne, T.J., Verstraete, F., 2006. Phys. Rev. Lett. 96, 220503.
Ostapchuk, D.C.M., Lin, S.Y., Mann, R.B., Hu, B.L., 2012. J. High Energy Phys. 07, 072.
Ostapchuk, D.C.M., Mann, R.B., 2009. Phys. Rev. A 79, 042333.
Pan, Q., Jing, J.L., 2008a. Phys. Rev. A 77, 024302.
Pan, Q., Jing, J.L., 2008b. Phys. Rev. D 78, 065015.
Pati, A.K., Wilde, M.M., Usha Devi, A.R., Rajagopal, A.K., Sudha, 2012. Phys. Rev. A 86, 042105.
Paul, T., Qureshi, T., 2017. Phys. Rev. A 95, 042110.
Paula, F.M., de Oliveira, T.R., Sarandy, M.S., 2013a. Phys. Rev. A 87, 064101.
Paula, F.M., Montealegre, J.D., Saguia, A., de Oliveira, T.R., Sarandy, M.S., 2013b. Europhys. Lett. 103, 50008.
Peng, Y., Jiang, Y., Fan, H., 2016. Phys. Rev. A 93, 032326.
Peres, A., Scudo, P.F., Terno, D.R., 2002. Phys. Rev. Lett. 88, 230402.
Peters, N.A., Altepeter, J.B., Branning, D., Jeffrey, E.R., Wei, T.-C., Kwiat, P.G., 2004. Phys. Rev. Lett. 92, 133601.
Petz, D., 2003. Rev. Mod. Phys. 15, 79.
Piani, M., 2012. Phys. Rev. A 86, 034101.
Piani, M., Cianciaruso, M., Bromley, T.R., Napoli, C., Johnston, N., Adesso, G., 2016. Phys. Rev. A 93, 042107.
Pierini, R., Moradi, S., Mancini, S., 2016. Int. J. Theor. Phys. 55, 3059.
Pino, M., 2014. Phys. Rev. B 90, 174204.
Pires, D.P., Céleri, L.C., Soares-Pinto, D.O., 2015. Phys. Rev. A 91, 042330.
Ponte, P., Papić, Z., Huveneers, F., Abanin, D.A., 2015. Phys. Rev. Lett. 114, 140401.
Popp, M., Verstraete, F., Martín-Delgado, M.A., Cirac, J.I., 2005. Phys. Rev. A 71, 042306.
Pozas-Kerstjens, A., Martin-Martinez, E., 2015. Phys. Rev. D 92, 064042.
Prodan, E., Hughes, T.L., Bernevig, B.A., 2010. Phys. Rev. Lett. 105, 115501.
Pusey, M.F., 2015. J. Opt. Soc. Am. B 32, A56.
Qi, X., Gao, T., Yan, F.L., 2017. J. Phys. A 50, 285301.
Qi, X.L., Katsura, H., Ludwig, A.W.W., 2012. Phys. Rev. Lett. 108, 196402.
Qiang, W.C., Zhang, L., 2015. Phys. Lett. B 742, 383.
Qiu, D., 2002. Phys. Lett. A 303, 140.
Qureshi, T., Siddiqui, M.A., 2017. Ann. Phys. 385, 598.
Radhakrishnan, C., Parthasarathy, M., Jambulingam, S., Byrnes, T., 2016. Phys. Rev. Lett. 116, 150504.
Ramzan, M., 2013. Quantum Inf. Process. 12, 2721.
Rana, S., Parashar, P., 2013. Quantum Inf. Process. 12, 2523.
Rana, S., Parashar, P., Lewenstein, M., 2016. Phys. Rev. A 93, 012110.
Rana, S., Parashar, P., Winter, A., Lewenstein, M., 2017. Phys. Rev. A 96, 052336.
Rastegin, A.E., 2016. Phys. Rev. A 93, 032136.
Refael, G., Moore, J.E., 2004. Phys. Rev. Lett. 93, 260602.
Regula, B., Lee, A.R., Dragan, A., Fuentes, I., 2016. Phys. Rev. D 93, 025034.
Richter, B., Omar, Y., 2015. Phys. Rev. A 92, 022334.
Rigol, M., Dunjko, V., Olshani, M., 2008. Nature 452, 854.
Roushan, P., Neill, C., Tangpanitanon, J., Bastidas, V.M., Megrant, A., Barends, R., Chen, Y., Chen, Z., Chiaro, B., Dunsworth, A., Fowler, A., Foxen, B., Giustina, M., Jeffrey, E., Kelly, J., Lucero, E., Mutus, J., Neeley, M., Quintana, C., Sank, D., Vainsencher, A., Wenner, J., White, T., Neven, H., Angelakis, D.G., Martinis, J., 2017. Science 358, 1175.
Sabín, C., Adesso, G., 2015. Phys. Rev. A 92, 042107.
Sabín, C., Fuentes, I., Johansson, G., 2015. Phys. Rev. A 92, 012314.
Sachdev, S., 2000. Quantum Phase Transitions. Cambridge University Press, Cambridge.
Samos-Sáenz de Buruaga, D.N., Sabín, C., 2017. Phys. Rev. A 95, 022307.
Sánchez-Ruiz, J., 1998. Phys. Lett. A 244, 189.
Satyabrata, A., Subhashish, B., 2012. Phys. Rev. A 86, 062313.
Shahandeh, F., Lund, A.P., Ralph, T.C., 2017. arXiv:1706.00478.
Shao, L.H., Xi, Z., Fan, H., Li, Y., 2015. Phys. Rev. A 91, 042120.
Shi, H.L., Liu, S.Y., Wang, X.H., Yang, W.L., Yang, Z.Y., Fan, H., 2017. Phys. Rev. A 95, 032307.
Shimony, A., 1995. Ann N.Y. Acad. Sci. 755, 675.
Silva, I.A., Souza, A.M., Bromley, T.R., Cianciaruso, M., Marx, R., Sarthour, R.S., Oliveira, I.S., Franco, R.L., Glaser, S.J., deAzevedo, E.R., Soares-Pinto, D.O., Adesso, G., 2016. Phys. Rev. Lett. 117, 160402.

Singh, U., Bera, M.N., Dhar, H.S., Pati, A.K., 2015. Phys. Rev. A 91, 052115.
Singh, U., Pati, A.K., Bera, M.N., 2016a. Mathematics 4, 47.
Singh, U., Zhang, L., Pati, A.K., 2016b. Phys. Rev. A 93, 032125.
Situ, H., Hu, X., 2016. Quantum Inf. Process. 15, 4649.
Skrzypczyk, P., Navascués, M., Cavalcanti, D., 2014. Phys. Rev. Lett. 112, 180404.
Smith, J., Lee, A., Richerme, P., Neyerhuis, B., Hess, P.W., Hauke, P., Heyl, M., Huse, D.A., Monroe, C., 2016. Nat. Phys. 12, 907.
Song, X.K., Wu, T., Xu, S., He, J., Ye, L., 2014. Ann. Phys. 349, 220.
Spehner, D., Orszag, M., 2013. New J. Phys. 15, 103001.
Spehner, D., Orszag, M., 2014. J. Phys. A 47, 035302.
Srednicki, M., 1994. Phys. Rev. E 50, 888.
Streltsov, A., Adesso, G., Piani, M., Bruß, D., 2012. Phys. Rev. Lett. 109, 050503.
Streltsov, A., Adesso, G., 2017a. Rev. Mod. Phys. 89, 041003.
Streltsov, A., Chitambar, E., Rana, S., Bera, M.N., Winter, A., Lewenstein, M., 2016. Phys. Rev. Lett. 116, 240405.
Streltsov, A., Kampermann, H., Bruß, D., 2010a. New J. Phys. 12, 123004.
Streltsov, A., Kampermann, H., Bruß, D., 2010b. New J. Phys. 12, 123004.
Streltsov, A., Kampermann, H., Bruß, D., 2011a. Phys. Rev. Lett. 107, 170502.
Streltsov, A., Kampermann, H., Bruß, D., 2011b. Phys. Rev. Lett. 106, 160401.
Streltsov, A., Kampermann, H., Wölk, S., Gessner, M., Bruß, D., 2018. New J. Phys. 20, 053058.
Streltsov, A., Rana, S., Boes, P., Eisert, J., 2017b. Phys. Rev. Lett. 119, 140402.
Streltsov, A., Singh, U., Dhar, H.S., Bera, M.N., Adesso, G., 2015. Phys. Rev. Lett. 115, 020403.
Styliaris, G., Venuti, L.C., Zanardi, P., 2018. Phys. Rev. A 97, 032304.
Tan, K.C., Kwon, H., Park, C.-Y., Jeong, H., 2016. Phys. Rev. A 94, 022329.
Tan, K.C., Volkoff, T., Kwon, H., Jeong, H., 2017. Phys. Rev. Lett. 119, 190405.
Terashima, H., 2000. Phys. Rev. D 61, 104016.
Terhal, B.M., Horodecki, P., 2000. Phys. Rev. A 61, 040301.
Theurer, T., Killoran, N., Egloff, D., Plenio, M.B., 2017. Phys. Rev. Lett. 119, 230401.
Tian, Z.H., Jing, J., 2013. Ann. Phys. 333, 76.
Tian, Z.H., Jing, J., 2014. Ann. Phys. 350, 1.
Titum, P., Lindner, N.H., Rechtsman, M.C., Refael, G., 2015. Phys. Rev. Lett. 114, 056801.
Tsallis, C., 1988. J. Stat. Phys. 52, 479.
Unruh, W.G., 1976. Phys. Rev. D 14, 870.
Unruh, W.G., Wald, R.M., 1984. Phys. Rev. D 29, 1047.
Vedral, V., Plenio, M.B., 1998. Phys. Rev. A 57, 1619.
Vedral, V., Plenio, M.B., Rippin, M.A., Knight, P.L., 1997. Phys. Rev. Lett. 78, 2275.
Verstraete, F., Audenaert, K., 2001. Phys. Rev. A 64, 012316.
Verstraete, F., Popp, M., 2004. Phys. Rev. Lett. 92, 027901.
Verstraete, F., Verschelde, H., 2002. Phys. Rev. A 66, 022307.
de Vicente, J.I., Streltsov, A., 2017. J. Phys. A 50, 045301.
Vidal, G., Latorre, J.I., Rico, E., Kitaev, A., 2003. Phys. Rev. Lett. 90, 227902.
Vidal, G., Tarrach, B., 1999. Phys. Rev. A 59, 141.
Vidal, G., Werner, R.F., 2002. Phys. Rev. A 65, 032314.
Virmani, S., Plenio, M., 2000. Phys. Lett. A 268, 31.
Žnidarič, M., Prosen, T., Prelovšek, P., 2008. Phys. Rev. B 77, 064426.
Wald, R.M., 1994. Quantum Field Theory in Curved Spacetimes and Black Hole Thermodynamics. The University of Chicago Press, Chicago.
Wang, J.C., Deng, J.F., 2010c. Phys. Rev. A 81, 052120.
Wang, J.C., Jing, J.L., 2010. Phys. Rev. A 82, 032324.
Wang, J.C., Jing, J.L., 2011. Phys. Rev. A 83, 022314.
Wang, J.C., Jing, J.L., 2012. Ann. Phys. 327, 283.
Wang, J.C., Jing, J.L., Fan, H., 2014b. Phys. Rev. D 90, 025032.
Wang, J.C., Jing, J.L., 2018a. Ann. Phys. 530, 1700261.
Wang, J.C., Pan, Q.Y., Chen, S.B., Jing, J.L., 2009. Phys. Lett. B 692, 202.
Wang, J.C., Pan, Q.Y., 2010a. Ann. Phys. 325, 1190.
Wang, J.C., Pan, Q.Y., 2010b. Phys. Lett. B 692, 202.
Wang, J.C., Tian, Z.H., Jing, J.L., Fan, H., 2016a. Phys. Rev. A 93, 062105.
Wang, J.C., Tian, Z.H., Jing, J.L., 2015. Nucl. Phys. B 892, 390.
Wang, Y.T., Tang, J.S., Wei, Z.Y., Yu, S., Ke, Z.J., Xu, X.Y., Li, C.F., Guo, G.C., 2017. Phys. Rev. Lett. 118, 020403.
Wang, Z., Wang, Y.L., Wang, Z.X., 2016c. Quantum Inf. Process. 15, 4641.
Wang, Z.A., Zeng, Y., Lang, H.F., Hong, Q.T., Cui, J., Fan, H., 2018b. arXiv:1802.08475.
Wei, T.C., Goldbart, P.M., 2003. Phys. Rev. A 68, 042307.
Werlang, T., Trippe, C., Ribeiro, G.A.P., Rigolin, G., 2010. Phys. Rev. Lett. 105, 095702.
Wigner, E.P., Yanase, M.M., 1963. Proc. Natl. Acad. Sci. U.S.A 49, 910.
Wilson, C.M., Johansson, G., Pourkabirian, A., Simoen, M., Johansson, J.R., Duty, T., Nori, F., Delsing, P., 2011. Nature 479, 376.
Winter, A., Yang, D., 2016. Phys. Rev. Lett. 116, 120404.
Wootters, W.K., 1998. Phys. Rev. Lett. 80, 2245.
Wootters, W.K., Fields, B.D., 1989. Ann. Phys. 191, 363.
Wootters, W.K., 1986. Found. Phys. 16, 391.
Wu, S., Poulsen, U.V., Mølmer, K., 2009. Phys. Rev. A 80, 032319.
Wu, S., Zhang, J., Yu, C., Song, H., 2014. Phys. Lett. A 378, 344–347.
Xi, Z., Li, Y., 2015. Sci. Rep. 5, 10922.
Xi, Z., Wang, X., 2012. Phys. Rev. A 85, 042325.
Xu, J., 2016. Phys. Rev. A 93, 032111.
Xu, J.S., Li, C.F., 2013. Int. J. Mod. Phys. B 27, 1345054.
Xu, K., Chen, J.J., Zeng, Y., Zhang, Y.R., Song, C., Liu, W.X., Guo, Q.J., Zhang, P.F., Xu, D., Deng, H., Huang, K.Q., Wang, H., Zhu, X.B., Zheng, D.N., Fan, H., 2018. Phys. Rev. Lett. 120, 050507.

Yang, C.N., 1989. Phys. Rev. Lett. 63, 2144.
Yao, Y., Dong, G.H., Ge, L., Li, M., Sun, C.P., 2016b. Phys. Rev. A 94, 062339.
Yao, Y., Dong, G.H., Xiao, X., Sun, C.P., 2016a. Sci. Rep. 6, 32010.
Yao, Y., Xiao, X., Ge, L., Sun, C.P., 2015. Phys. Rev. A 92, 022112.
You, B., Cen, L.X., 2012. Phys. Rev. A 86, 012102.
Yu, C.S., 2017. Phys. Rev. A 95, 042337.
Yu, C.S., Wu, S.X., Wang, X., Yi, X.X., Song, H.S., 2014. Europhys. Lett. 107, 10007.
Yu, C.S., Yang, S.R., Guo, B.Q., 2016a. Quantum Inf. Process. 15, 3773.
Yu, X.D., Zhang, D.J., Liu, C.L., Tong, D.M., 2016b. Phys. Rev. A 93, 060303.
Yu, X.D., Zhang, D.J., Xu, G.F., Tong, D.M., 2016c. Phys. Rev. A 94, 060302.
Yuan, X., Bai, G., Peng, T., Ma, X., 2017. Phys. Rev. A 96, 032313.
Yuan, X., Zhou, H., Cao, Z., Ma, X., 2015. Phys. Rev. A 92, 022124.
Zanardi, P., Styliaris, G., Venuti, L.C., 2017. Phys. Rev. A 95, 052306.
Zeng, B., Chen, X., Zhou, D.L., Wen, X.G., 2015. arXiv:1508.02595.
Zhang, D.J., Liu, C.L., Yu, X.D., Tong, D.M., 2018. Phys. Rev. Lett. 120, 170501.
Zhang, H.J., Chen, B., Li, M., Fei, S.M., Long, G.L., 2017a. Commun. Theor. Phys. 67, 166.
Zhang, J., Yang, S.R., Zhang, Y., Yu, C.S., 2017b. Sci. Rep. 7, 45598.
Zhang, L., 2017. J. Phys. A 50, 155303.
Zhang, S., Feng, Y., Sun, X., Ying, M., 2001. Phys. Rev. A 64, 062103.
Zhang, Y.J., Han, W., Xia, Y.J., Yu, Y.M., Fan, H., 2015. Sci. Rep. 5, 13359.
Zhang, Y.R., Shao, L.H., Li, Y., Fan, H., 2016. Phys. Rev. A 93, 012334.
Zhang, L., Singh, U., Pati, A.K., 2017c. Ann. Phys. 377, 125.
Zhao, Q., Liu, Y., Yuan, X., Chitambar, E., Ma, X., 2018. Phys. Rev. Lett. 120, 070403.
Zurek, W.H., 2003. Phys. Rev. A 67, 012320.